BERNSTEIN POLYNOMIALS FOR FUNCTIONS OF TWO VARIABLES OF CLASS \( C^{(k)} \)

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Introduction. Let \( \phi(x, y) \) be a continuous function of the real variables \( x \) and \( y \), where \( x \) and \( y \) are in the closed region \( R: 0 \leq x \leq 1, \ 0 \leq y \leq 1 \). The Bernstein polynomial \( B_{mn}(x, y) \) associated with the function \( \phi(x, y) \) is defined as

\[
B_{mn}(x, y) = \sum_{p=0}^{n} \sum_{q=0}^{m} \phi \left( \frac{p}{n}, \frac{q}{m} \right) \lambda_{n,p}(x) \lambda_{m,q}(y)
\]

where \( \lambda_{n,p}(x) = C_{n,p} x^p (1-x)^{n-p}, \lambda_{m,q}(y) = C_{m,q} y^q (1-y)^{m-q} \).

A function \( \phi(x, y) \) is said to be of class \( C^{(k)} \) for \( x \) and \( y \) in \( R \), if the partial derivatives of order \( 1, 2, \cdots, k \) of \( \phi(x, y) \) exist and are continuous. We shall use the notation

\[
\phi^{(i,k-i)} = \frac{\partial^{(k)} \phi(x, y)}{\partial x^i \partial y^{k-i}} \quad (i = 0, 1, \cdots, k)
\]

and, for brevity, shall omit functional arguments from expressions whenever possible.

It is the purpose of this paper to prove the

Theorem. If \( \phi(x, y) \) is of class \( C^{(k)} \) for \( x \) and \( y \) in \( R \), then

\[
\lim_{m,n \to \infty} B^{(i,k-i)}_{mn} = \phi^{(i,k-i)}
\]

and the convergence is uniform in \( R \).

This theorem, for functions of one variable and for \( k = 0 \), was proved by S. Bernstein [1],2 and again for functions of one variable but for arbitrary \( k \), the theorem was proved by S. Wigert [2]. The process of extending the results of Bernstein and Wigert to functions of two variables of class \( C^{(k)} \) introduces aspects which are of interest.

Preliminary results. We shall make use of the relations

\[
\sum_{p=0}^{n} \lambda_{n,p}(x) = 1,
\]

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2 Numbers in brackets refer to the bibliography at the end of the paper.

64
If we define, for $k \geq 0$, $i = 0, 1, \ldots, k$,

$$A^{(i,k-i)}_{p,q} = \sum_{a=0}^{i} \sum_{b=0}^{k-i} (-1)^{a+b} C_{i,a} C_{k-i,b} \left( \frac{p + (i - \alpha)}{n}, \frac{q + (k - i - \beta)}{m} \right),$$

then, by mathematical induction, the following two lemmas can be established.

**Lemma 1.** If $0 \leq i \leq k$, $i \leq n$, $k \leq m$, $x$ and $y$ in $R$, then the $k$th partial derivatives of the Bernstein polynomials (1) are given by

$$B^{(i,k-i)}_{mn} = \frac{n!m!}{(n-i)!(m-k+i)!} \sum_{p=0}^{n-i} \sum_{q=0}^{m-k+i} A^{(i,k-i)}_{p,q} \lambda_{n-i,p} \lambda_{m-k+i,q}.$$

**Lemma 2.** If $0 \leq i \leq k$, $0 \leq p \leq n-i$, $0 \leq q \leq m-k+i$, and if $\phi(x, y)$ is of class $C^{(k)}$ for $x$ and $y$ in $R$, then there exist two real numbers $\xi = \xi(p)$, $\gamma = \gamma(q)$ such that $0 < \xi < 1$, $0 < \gamma < 1$ and such that

$$A^{(i,k-i)}_{p,q} = \frac{1}{n!m^{k-i}} \phi^{(i,k-i)} \left( \frac{p + \xi}{n}, \frac{q + \gamma(k - i)}{m} \right).$$

The next lemma is basic for the proof of the theorem.

** Lemma 3.** For fixed $x$ and $y$ in $R$ and for fixed positive integers $M$ and $N$, let $d$ be an arbitrary positive number and let $a(p, q)$ be a quantity dependent upon $p$ and $q$ and such that

$$|a(p, q)| \leq \pi_1 \text{ for } \left| x - \frac{p}{N} \right| \leq d \text{ and } \left| y - \frac{q}{M} \right| \leq d,$$

$$|a(p, q)| \leq \pi_2 \text{ for } \left| x - \frac{p}{N} \right| > d \text{ and } \left| y - \frac{q}{M} \right| > d.$$

Furthermore, assume that it is possible to split off from $a(p, q)$ terms $a'(p)$ independent of $q$ or terms $b'(q)$ independent of $p$, that is,

$$a(p, q) = a''(p, q) + a'(p) = b''(p, q) + b'(q)$$

such that

$$|a''(p)| \leq \pi_3 \text{ for } \left| x - \frac{p}{N} \right| \leq d \text{ and } \left| y - \frac{q}{M} \right| > d,$$

$$|a'(p, q)| \leq \pi_4 \text{ for } \left| x - \frac{p}{N} \right| \leq d,$$

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\[ b''(p, q) \leq \pi_s \text{ for } \left| x - \frac{p}{N} \right| > d \] and \[ \left| y - \frac{q}{M} \right| \leq d, \]

then
\[
\sum_{p=0}^{N} \sum_{q=0}^{M} a(p, q)\lambda_{N,p}\lambda_{M,q} \leq \pi_1 + \pi_4 + \pi_8 + \frac{\pi_2(M + N)}{8MN^2d^2} + \frac{\pi_5}{4M^2d^2} + \frac{\pi_8}{4Nd^2}.
\]

**Proof.** Consider the inequality
\[
\sum_{p=0}^{N} \sum_{q=0}^{M} a(p, q)\lambda_{N,p}\lambda_{M,q} \leq \sum_{|x-p/N| \leq d} \sum_{|y-q/M| \leq d} |a(p, q)| \lambda_{N,p}\lambda_{M,q} + \sum_{|x-p/N| > d} \sum_{|y-q/M| > d} |a''(p, q) + a'(p)| \lambda_{N,p} + \lambda_{M,q} + \sum_{|x-p/N| > d} \sum_{|y-q/M| > d} |b''(p, q) + b'(q)| \lambda_{N,p}\lambda_{M,q}
\]

\[ = s_1 + s_2 + s_3 + s_4, \]

then
\[ s_1 \leq \sum_{|x-p/N| \leq d} \sum_{|y-q/M| \leq d} \lambda_{N,p}\lambda_{M,q} \leq \pi_1 \sum_{p=0}^{N} \sum_{q=0}^{M} \lambda_{N,p}\lambda_{M,q} = \pi_1 \]

and hence \( s_1 \leq \pi_1. \)

The inequalities in (5) imply the inequality
\[
\frac{M^2(Mx - p)^2 + N^2(My - q)^2}{2M^2N^2d^2} > 1
\]

and by (5) again
\[
|a(p, q)| \leq \pi_2 < \frac{\pi_2}{2M^2N^2d^2} \{M^2(Nx - p)^2 + N^2(My - q)^2\}.
\]
thus

\[ S_2 \leq \frac{\pi_2}{2M^2N^2d^2} \sum_{p=0}^{N} \sum_{q=0}^{M} \left\{ M^2(Nx - p)^2 + N^2(My - q)^2 \right\} \lambda_{N,p} \lambda_{M,q} \]

\[ = \frac{\pi_2}{2MN^2d^2} \left[ Mx(1 - x) + Ny(1 - y) \right] \]

(by (3) and (2))

and since

\[ \max_{x,y \in R} \left[ Mx(1 - x) + Ny(1 - y) \right] = \frac{M + N}{4} , \]

we have

\[ S_2 \leq \frac{\pi_2(M + N)}{8MN^2d^2} . \]

In similar fashion, using (6), (7), (8), and (9), we obtain

\[ S_3 \leq \frac{\pi_3}{4Md^2} , \quad S_4 \leq \frac{\pi_5}{4Nd^2} . \]

Collecting these estimates in (11) we obtain (10).

**Proof of the theorem.** By Lemmas 1 and 2, we have the inequality

\[ |B_{mn}^{(i,k-\ell)} - \phi^{(i,k-\ell)}| \]

\[ \leq \left| \sum_{p=0}^{n-i} \sum_{q=0}^{m-k+i} \left\{ 1 - \frac{n!m!}{(n-i)!((m-k+i)!n_m^m k-i)!} \right\} \phi^{(i,k-\ell)} \left( \frac{p + \xi i}{n}, \frac{q + \gamma(k - i)}{m} \right) \lambda_{n-i,p} \lambda_{m-k+i,q} \right| 

\[ + \sum_{p=0}^{n-i} \sum_{q=0}^{m-k+i} \left\{ \phi^{(i,k-\ell)} \left( \frac{p + \xi i}{n}, \frac{q + k(k - i)}{m} \right) \right\} \lambda_{n-i,p} \lambda_{m-k+i,q} \right| = S_1 + S_2. \]

We first estimate \( S_2 \). Let \( \phi^{(i,k-\ell)}(x, y) = \psi(x, y) \), then since \( \psi(x, y) \) is continuous for \( x \) and \( y \) in \( R \), it is uniformly continuous and bounded on \( R \). That is, for an arbitrary positive number \( \epsilon \), there exists a number \( d(\epsilon) > 0 \) such that if \( |x - x_1| \leq d(\epsilon)/2 \) and \( |y - y_1| \leq d(\epsilon)/2 \), where \( x_1 \) and \( y_1 \) are in \( R \), then

\[ |\psi(x_1, y_1) - \psi(x, y)| \leq \epsilon/6 \]

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and

\[ |\psi(x, y)| \leq L^{(k)}, \quad \text{for } x \text{ and } y \text{ in } \mathbb{R}, \]

where \( L^{(k)} = \max_{x,y \in \mathbb{R}} |\psi(x, y)|. \)

If \( p, n, i \) are positive integers such that \( 0 \leq p \leq n - i \), and if \( \xi \) is a real number such that \( 0 < \xi < 1 \), then

\[ \left| \frac{p}{n - i} - \frac{p + \xi i}{n} \right| < \frac{i}{n}, \]

and if \( m, q, k, i \) are positive integers such that \( 0 \leq q \leq m - (k - i) \), and \( \gamma \) a real number such that \( 0 < \gamma < 1 \), then

\[ \left| \frac{q}{m - k + i} - \frac{q + \gamma(k - i)}{m} \right| < \frac{k - i}{m}. \]

Now if \( x \) is fixed and \( P \) is such that \( 0 \leq P \leq n - 1 \) and \( |x - P/(n - i)| \leq d(e)/2 \), then by (14)

\[ \left| x - \frac{P + \xi i}{n} \right| \leq \frac{d(e)}{2} + \frac{i}{n} \]

and if we choose

\[ N_{1x} > 2i/d(e) + i, \]

then for \( n > N_{1x} \)

\[ i/n < d(e)/2 \]

and

\[ \left| x - \frac{P + \xi i}{n} \right| \leq d(e) \quad \text{if } n > N_{1x}. \]

In the same manner, choose

\[ M_{1x} > (k - i) + \frac{2(k - i)}{d(e)}; \]

then for fixed \( y \) and \( q \) such that \( 0 \leq q \leq m - k + i \) and \( |y - q/(m - k + i)| \leq d(e)/2 \), we obtain from (15)

\[ \left| y - \frac{q + \gamma(k - i)}{m} \right| \leq d(e) \quad \text{if } m > M_{1x}. \]

If we choose \( x_1 = (P + \xi i)/n \), \( y_1 = (q + \gamma(k - i))/m \) in (12), then from (16), (17) we have

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BERNSTEIN POLYNOMIALS

(18) \[ |\psi\left(\frac{p + \xi i}{n}, \frac{q + \gamma(k - i)}{m}\right) - \psi(x, y)| \leq \frac{e}{6} \]

if \( n > N_{1n}, \ m > M_{1m}, |x - (p + \xi i)/n| \leq d(\varepsilon)/2, \ |y - (q + \gamma(k - i))/m| \leq d(\varepsilon)/2. \)

If we now let:
- \( \Delta_{n-i} \) indicate summation for all \( p \) such that \( |x - p/(n - i)| \leq d(\varepsilon)/2, \)
- \( \Delta'_{n-i} \) indicate summation for all \( p \) such that \( |x - p/(n - i)| > d(\varepsilon)/2, \)
- \( \Delta_{m-k+i} \) indicate summation for all \( q \) such that \( |y - q/(m - k + i)| \leq d(\varepsilon)/2, \)
- \( \Delta'_{m-k+i} \) indicate summation for all \( q \) such that \( |y - q/(m - k + i)| > d(\varepsilon)/2, \)

then \( S_2 \) may be written as

\[
S_2 \leq \sum_{\Delta_{n-i}} \sum_{\Delta_{m-k+i}} \psi\left(\frac{p + \xi i}{n}, \frac{q + \gamma(k - i)}{m}\right) - \psi(x, y) \lambda_{n-i, p, m-k+i, q} \\
+ \sum_{\Delta'_{n-i}} \sum_{\Delta'_{m-k+i}} \psi\left(\frac{p + \xi i}{n}, \frac{q + \gamma(k - i)}{m}\right) - \psi(x, y) \lambda_{n-i, p, m-k+i, q} \\
+ \sum_{\Delta_{n-i}} \sum_{\Delta_{m-k+i}} \psi\left(\frac{p + \xi i}{n}, \frac{q + \gamma(k - i)}{m}\right) - \psi\left(\frac{p + \xi i}{n}, y\right) \lambda_{n-i, p, m-k+i, q} \\
+ \sum_{\Delta'_{n-i}} \sum_{\Delta'_{m-k+i}} \psi\left(\frac{p + \xi i}{n}, \frac{q + \gamma(k - i)}{m}\right) - \psi\left(x, \frac{q + \gamma(k - i)}{m}\right) \lambda_{n-i, p, m-k+i, q} \\
+ \sum_{\Delta_{n-i}} \sum_{\Delta_{m-k+i}} \psi\left(x, \frac{q + \gamma(k - i)}{m}\right) - \psi(x, y) \lambda_{n-i, p, m-k+i, q} \\
+ \sum_{\Delta'_{n-i}} \sum_{\Delta'_{m-k+i}} \psi\left(x, \frac{q + \gamma(k - i)}{m}\right) - \psi(x, y) \lambda_{n-i, p, m-k+i, q}.
\]
If we use (13) and (18) and let
\[ a(p, q) = \psi\left(\frac{p + \xi i}{n}, \frac{q + \gamma (k - i)}{m}\right) - \psi(x, y), \]
\[ d = d(e)/2, \]
\[ \pi_1 = \pi_4 = \pi_6 = e/6, \]
\[ \pi_2 = \pi_3 = \pi_5 = 2L^{(e)}, \]
then the hypotheses of Lemma 3 are satisfied and (19) becomes
\[ S_2 \leq \frac{e}{2} + \frac{3L^{(e)}}{[d(e)]^2} \left( \frac{1}{n - i} + \frac{1}{m - k + i} \right), \quad \text{for } n > N_{1e}, m > M_{1e}. \]

To estimate \( S_1 \), we note that there exist two numbers \( N_{2e} > i, M_{2e} > k - i \) such that if \( n > N_{2e}, m > M_{2e} \), then
\[ \left| 1 - \frac{n!m!}{(n - i)!(m - k + i)!} \right| < \frac{e}{6L^{(e)}}. \]

Thus
\[ S_1 \leq \frac{e}{6L^{(e)}} \sum_{p=0}^{n-i} \sum_{q=0}^{m-k+i} \psi\left(\frac{p + \xi i}{n}, \frac{q + \gamma (k - i)}{m}\right) |\lambda_{n-i, p}\lambda_{m-k+i, q}| \]
\[ \leq \frac{e}{6} \sum_{p=0}^{n-i} \sum_{q=0}^{m-k+i} \lambda_{n-i, p}\lambda_{m-k+i, q}, \]
and hence
\[ S_1 \leq e/6. \]

Using these estimates of \( S_1 \) and \( S_2 \) we have
\[ |B^{(k)}_{mn} - \phi^{(i, k-i)}| \leq \frac{2}{3} e + \frac{3L^{(e)}}{[d(e)]^2} \left[ \frac{1}{n - i} + \frac{1}{m - k + i} \right] \]
for \( n > N_{1e}, N_{2e}; m > M_{1e}, M_{2e} \).

Let
\[ N_{2e} > i + \frac{18L^{(e)}}{e[d(e)]^2}, \]
\[ M_{2e} > (k - i) + \frac{18L^{(e)}}{e[d(e)]^2}, \]
and take
\[ N_\varepsilon = \max (N_1 \varepsilon, N_2 \varepsilon, N_3 \varepsilon), \]
\[ M_\varepsilon = \max (M_1 \varepsilon, M_2 \varepsilon, M_3 \varepsilon). \]

In (21) let \( n > N_\varepsilon \) and such that
\[ 3L^{(k)} \left[ \frac{1}{d(\varepsilon)} \right]^2(n - i) < \frac{\varepsilon}{6}, \]
and in (22) let \( m > M_\varepsilon \) and such that
\[ 3L^{(k)} \left[ \frac{1}{d(\varepsilon)} \right]^2(m - k + i) < \frac{\varepsilon}{6}. \]

Thus, for \( n > N_\varepsilon \) and \( m > M_\varepsilon \), (20) becomes
\[ \left| B^{(i, k-i)}_{m, i} - \phi^{(i, k-i)} \right| < \varepsilon \]
and the theorem is proved.

**Bibliography**