REMARKS ON A THEOREM OF E. J. McSHANE

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In a recent paper E. J. McShane [3] has given a theorem which is the common core of a variety of results about Baire sets, Baire functions, and convex sets in topological spaces including groups and linear spaces. In general terms his theorem states that if \( \mathcal{J} \) is a family of open maps defined in one topological space \( X_1 \) into another, \( X_2 \), the total image \( \mathcal{J}(S) \) of a second category Baire set \( S \) in \( X_1 \) has, under certain conditions on \( \mathcal{J} \) and \( S \), a nonvacuous interior. The point of these remarks is to show that his argument yields a theorem for a larger class than the second category Baire sets. From this there follow slightly stronger and more specific versions of some of his results, including his principal theorem, as well as a proof that if \( S \) is a subset of a weak sort of topological group and \( S \) contains a second category Baire set, then the identity element lies in the interior of both \( S^{-1}S \) and \( SS^{-1} \). There is also at the end an extension of Zorn's theorem on the structure of certain semigroups.

In a topological space \( X \) let the closure and interior of a set \( E \) be denoted by \( E^* \) and \( E^o \) and the null set by \( \Lambda \). For any set \( S \) let \( I(S) = \bigcup \{ G \mid C^o \text{ open, } G \cap S \text{ is first category} \} \) and \( II(S) = X - I(S) \), and let \( III(S) \) be the open set \( II(S)^o \cap I(X - S) \). By a fundamental theorem of Banach [2], \( S \cap I(S)^* \) is first category and hence \( S \) is second category if and only if \( II(S)^o \neq \Lambda \). From these we note that if \( N \) is a non-null open subset of \( III(S) \), then \( N - S \) is in the first category set \( I(X - S) \), and \( N \cap S \) cannot be first category since \( N \) is non-null open and disjoint with \( I(S) \). This gives us the following lemma.

**Lemma 1.** For any non-null open subset \( N \) of \( III(S) \), the sets \( N - S \) and \( N \cap S \) are first and second category respectively.

We recall that \( S \) is defined to be a Baire set in \( X \) if \( (S - G) \cup (G - S) \) is first category for some open set \( G \); an equivalent condition is \( II(S)^o \subset I(X - S) \), or \( II(S)^o = III(S) \).

**Definition.** \( \mathfrak{B}(X) \) is the family of all second category Baire sets in \( X \). A set \( S \) is in \( \mathfrak{A}(X) \) if and only if \( S \) contains an element of \( \mathfrak{B}(X) \).

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Any member \( S \) of \( \mathfrak{A}(X) \) is characterized by \( I(X-S) \supset II(S)^\circ \neq \Lambda \) or by \( III(S) = II(S)^\circ \neq \Lambda \). A characterization of the class \( \mathfrak{A}(X) \), which is generally larger (for example, when \( X \) is the reals), is given by the following lemma.

**Lemma 2.** \( S \in \mathfrak{A}(X) \) is equivalent to each of these: (1) \( III(S) \neq \Lambda \), (2) there is a non-null open set \( N \) such that \( N - S \) is first category and \( N \cap S \) is second category.

If \( S \in \mathfrak{A}(X) \), let \( B \in \mathfrak{A}(X) \) with \( B \subseteq S \). The latter implies \( II(S)^\circ \supset II(B)^\circ \) and \( I(X-S) \supset I(X-B) \) and hence \( III(S) \supset III(B) \neq \Lambda \). When (1) holds, the set \( N = III(S) \) satisfies (2) by Lemma 1 above. When (2) is true, let \( B = N \cap S \). Then \( B \) is second category and differs from the open set \( N \) only on the first category set \( N - S \); thus \( B \in \mathfrak{A}(X) \).

Now let \( X_1 \) and \( X_2 \) be topological spaces and \( \mathcal{J} = \{ f \} \) a non-null family of functions defined in \( X_1 \) to \( X_2 \) where the domain of definition \( D_f \) of each \( f \) is open in \( X_1 \) and each \( f \) is open on its domain to \( X_2 \), that is, maps open sets into open sets. For any sets \( E_1 \subseteq X_1 \) and \( E_2 \subseteq X_2 \) we shall write \( \mathcal{J}(E_1) \) for the set \( \bigcup \{ f(E_1 \cap D_f) \mid f \in \mathcal{J} \} \) and \( \mathcal{J}^{-1}(E_2) \) for the set of all \( x \in X_1 \) such that \( f(x) \) is defined and in \( E_2 \) for some \( f \in \mathcal{J} \).

For any pair of sets \( S_0 \) and \( N \) in \( X_1 \) consider these conditions on \( S_0, N, \) and \( \mathcal{J} \): (i) \( S_0 \neq \Lambda \) and \( S_0 \subseteq D_f \) for each \( f \in \mathcal{J} \); (ii) \( N \) is open and \( N \supset S_0 \); (iii) \( f(N \cap D_f) \subseteq \mathcal{J}(S_0) \) for each \( f \in \mathcal{J} \). We note that (iii) is equivalent to the following: (iii') \( y \in \mathcal{J}(N) \) implies \( S_0 \cap \mathcal{J}^{-1}(y) \neq \Lambda \). The essential proposition in McShane's theorem can be stated as follows.

**Lemma 3.** When \( \mathcal{J}, S_0, \) and \( N \) satisfy (i), (ii), and (iii), \( \mathcal{J}(S_0) \) is non-null and open.

For (following McShane) if \( U = \bigcup f(N \cap D_f) \), then from the original assumptions on \( \mathcal{J} \) clearly \( U \) is open in \( X_2 \), and \( U \subseteq \mathcal{J}(S_0) \) from (iii). By (i) and (ii) on the other hand, \( S_0 \subseteq D_f \cap N \) for each \( f \) and hence \( \mathcal{J}(S_0) \subseteq U \). And \( \mathcal{J}(S_0) \neq \Lambda \) since \( S_0 \) and \( \mathcal{J} \) are nonvacuous and \( S_0 \subseteq D_f \) for all \( f \).

Now suppose these are true for two sets \( N \) and \( S \) in \( X_1 \): (i') \( \Lambda \neq N \cap S \subseteq D_f \) for all \( f \) in \( \mathcal{J} \); (ii') \( N \) is open. If we set \( S_0 = N \cap S \) clearly (i) and (ii) hold; hence we have the following lemma.

**Lemma 4.** If \( \mathcal{J}, S, \) and \( N \) satisfy (i') and (ii') and also either (iii) or (iii') with \( S_0 = N \cap S \), then \( \mathcal{J}(N \cap S) \) is a non-null open set.

Obviously (iii') is true for \( S_0 = N \cap S \) if \( N \cap S \cap \mathcal{J}^{-1}(y) \) is second
category for each \( y \in \mathcal{J}(N) \); and the latter clearly holds whenever \( N - S \) is first category and the following is true: (iii’’) \( y \in \mathcal{J}(N) \) implies \( N \cap \mathcal{J}^{-1}(y) \) is second category. Lemma 4 now gives us the following lemma.

**Lemma 5.** \( \mathcal{J}(N \cap S) \) is non-null and open whenever \( \mathcal{J} \), \( S \), and \( N \) satisfy (i’), (ii’), and (iii’’) and \( N - S \) is first category.

From Lemmas 1 and 5 we have immediately the following theorem.

**Theorem.** \( \mathcal{J}(N \cap S) \) is non-null and open provided that \( N \) is a non-null open subset of \( III(S) \), that \( D_f \supseteq S \cap N \) for each \( f \) in \( \mathcal{J} \), and that (iii’’) is satisfied.

In view of Lemma 2 the theorem is concerned with and only with elements \( S \) of \( \mathfrak{A}(X_1) \). When \( S \) is in the more restricted class \( \mathfrak{B}(X_1) \), that is, when \( III(S) = II(S)^0 \neq \Lambda \), the following slightly sharper version of McShane’s theorem results on taking \( N \) to be \( II(S)^0 \).

**Corollary 1.** If \( S \in \mathfrak{B}(X_1) \), if \( D_f \supseteq S \cap II(S)^0 \) for each \( f \) in \( \mathcal{J} \), and if \( II(S)^0 \cap \mathcal{J}^{-1}(y) \) is second category for each \( y \in \mathcal{J}(N) \), then \( \mathcal{J}(S) = \mathcal{J}(S \cap II(S)^0) \cup \mathcal{J}(S \cap II(S)^0)^* \) where \( \mathcal{J}(S \cap II(S)^0) \) is non-null and open in \( X_2 \) and \( S \cap II(S)^0 \) is first category in \( X_1 \).

Now let \( X \) be a group having a topology in which \( xy \) is continuous in each variable and let \( e \) be the identity element. We recall the following properties of the function \( II(E) \) in any topological space [2, pp. 46–47]: (a) \( II(E) \) is always closed, \( II(II(E)) = II(E) \), and \( II(E) \subseteq II(F) \) when \( E \subseteq F \); (β) \( II(E)^0 \cap E^* = II(E) \cap E^* \); (γ) for any open set \( G \), \( II(G \cap II(E)^0) \supseteq G \cap II(E) \); and (δ) for any homeomorphism \( \phi \) in \( X \), \( II(\phi(E)) = \phi(II(E)) \).

**Corollary 2.** Let \( R \) be second category in \( X \) and \( S \subseteq \mathfrak{A}(X) \). Suppose \( G \) and \( H \) are open and \( G \cap II(R) \neq H \cap III(S) \). If we set \( A = G \cap R \cap II(R) \) and \( B = H \cap S \cap III(S) \), it follows that \( A^{-1}B \) and \( BA^{-1} \) are non-null open subsets of \( R^{-1}S \) and \( SR^{-1} \) respectively.

For each \( a \in A \) define \( f_a(x) = a^{-1}x \) for all \( x \) in the non-null open set \( N = H \cap III(S) \) and set \( \mathcal{J} = \{ f_a \} \). Since \( G \cap II(R) \neq \Lambda \) it follows from (γ) above that \( A \neq \Lambda \). Thus \( \mathcal{J} \) is non-null, each \( f_a \) is open on its open domain, and \( D_{f_a} \supseteq S \cap N = B \) for each \( f_a \). If it is shown that \( N \cap \mathcal{J}^{-1}(y) \) is second category whenever \( y \in \mathcal{J}(N) \), the theorem is applicable and \( \mathcal{J}(N \cap S) = \mathcal{J}(B) = A^{-1}B \) is a non-null open subset of \( R^{-1}S \). If \( y \in \mathcal{J}(N) \), then \( y = a^{-1}x \) for some \( a \in A \) and \( x \in N \), and \( \mathcal{J}^{-1}(y) = Aa^{-1}x \). Hence \( II(\mathcal{J}^{-1}(y)) = II(Aa^{-1}x) = II(A)A^{-1}x \) by (δ) above; since \( II(A) \supseteq A \) by (γ), it follows that \( II(\mathcal{J}^{-1}(y)) \ni x \). Then, since \( N \ni x \) and is open,
$N \cap \gamma^{-1}(y)$ must be second category. A similar proof, setting $f_*(x) = xa^{-1}$, establishes the theorem’s other assertion.

Taking $G = H = X$ we have the following corollaries.

**Corollary 3.** Let $S \subseteq \mathbb{A}(X)$. If $R$ is second category then $[R \cap II(R)^{-1}] [S \cap III(S)]$ and $[S \cap III(S)] [R \cap II(R)^{-1}]$ are non-null open subsets of $R^{-1}S$ and $SR^{-1}$ respectively.

**Corollary 4.** If $S \subseteq \mathbb{A}(X)$ and $R^{-1}$ is second category the sets $RS$ and $SR$ have non-null interiors.

**Corollary 5.** When $S \subseteq \mathbb{A}(X)$ it follows that $e \in (S^{-1}S)^0 \cap (SS^{-1})^0$.

**Corollary 6.** If $S$ is a subgroup and $S \subseteq \mathbb{A}(X)$, then $S = S^0$ and hence, since $S$ is a subgroup, $S = S^*$.

Corollaries 4 and 6 are slight extensions of results of McShane and Banach [3]. Corollary 5 for a more restricted $X$ is in another paper [4].

**Corollary 7.** Suppose $E^{-1}$ is second category in $X$ whenever $E$ is second category. If $R$ is second category and $S \subseteq \mathbb{A}(X)$, then

1. $[R \cap II(R^{-1})^{-1}] [S \cap III(S)]$ and $[S \cap III(S)] [R \cap II(R^{-1})^{-1}]$ are non-null open subsets of $RS$ and $SR$ respectively, and

2. $II(R) III(S)^* \subset ((R^0)^* \cap III(S))^*$ and $III(S)^* II(R) \subset (SR)^0 *$.

Conclusion (1) results immediately from Corollary 3 since $R^{-1}$ is second category. To establish (2) let $C = R \cap II(R^{-1})^{-1}$ and $D = S \cap III(S)$, and note that (1) implies that $C*D^* \subset (CD)^* \subset ((RS)^0)^*$ and $D^*C^* \subset ((SR)^0)^*$. It is thus sufficient to show that $II(R) \subset C^*$ and $III(S) \subset D^*$. The latter follows from (γ) above; for on setting $G = I(X - S)$ and $E = S$ therein we have $II(D) \subset III(S)$ and hence $D^* \subset III(S)$. For the former, consider any open set $N$ intersecting $II(R)$. The set $N \cap R$ is second category and hence $(N \cap R)^{-1}$ is a second category subset of $R^{-1}$. From this it follows that $(N \cap R)^{-1} \cap II(R^{-1}) \neq \Lambda$, for otherwise $(N \cap R)^{-1}$ is in the first category set $R^{-1} \cap I(R^{-1})$. Taking inverses we have $\Lambda \neq N \cap R \cap II(R^{-1})^{-1} = N \cap C$, proving that $II(R) \subset C^*$.

When $S \subseteq \mathbb{A}(X)$ the terms $III(S)$ and $III(S)^*$ in Corollary 7 can be replaced by $II(S)^0$ and $II(S)$, and hence in particular $II(S)^2 \subset ((S^{1})^{0})^*$. If also $S$ is a semigroup, that is, $S^2 \subset S$, then $II(S)^2 \subset (S^{0})^*$. It may also be remarked that the first assumption in Corollary 7 is weaker than that of assuming $x^{-1}$ to be continuous in $x$, as is shown by the reals with intervals $a \leq x < b$ as neighborhoods.
Lemma 6. For any sets $R$ and $S$ in $X$ let $\Gamma(R, S) = II(R)S^* \cup R^*II(S) \cup II(R)II(S^*) \cup II(R^*)II(S)$ and $\Delta(R, S) = II(R)S^* \cup R^*II(S) \cup II(R)II(S^*)$. Then $(RS)^* \supseteq II(RS) \supseteq \Gamma(R, S) \supseteq \Delta(R, S) \supseteq II(R)II(S)$.

For any $s \in S$ and $r \in R$ we have $II(R)s = II(Rs) \supseteq II(RS)$ and $rII(S) = II(rs) \supseteq II(RS)$, so that $II(R)S \cup RII(S) \supseteq II(RS)$. Since $II(RS)$ is closed and $A*B \subset (AB)^*$ for any $A$ and $B$, it follows that $II(R)S^* \cup R^*II(S) \subset II(RS)$. Moreover $II(RS) = II(RII(S)) \supseteq II(RII(S^*)) \cup II(R^*II(S))$. But from what has already been shown we have $II(RII(S^*)) \supseteq II(II(R))S^* \cup II(R)*II(S^*) = II(R)S^* \cup R^*II(S)$, and similarly $II(R^*II(S)) \supseteq II(R^*)II(S) \cup R^*II(S)$. Thus $II(RS) \supseteq \Gamma(R, S)$. The rest is obvious.

From this it is clear that $II(S^2) \supseteq \Gamma(S, S) \supseteq \Delta(S, S) \supseteq II(S)^2$ for any $S$. Another consequence is the following corollary.

Corollary 8. Suppose $E^{-1}$ is second category whenever $E$ is second category. If $R$ is second category, $S \in B\mathfrak{B}(X)$, and $S \supseteq R \cup S$, then

$$(S^o)^* \supseteq II(R) [\Gamma(R, S) \cup \Gamma(S, R)] \cup [\Gamma(R, S) \cup \Gamma(S, R)] \cup R \supseteq II(R).$$

Clearly $(S^o)^* \supseteq (RS)^o)^*$, and from a remark after Corollary 7,

$$(RS)^o \supseteq II(R)II(S).$$

At the same time, obviously $II(S) \supseteq II(RS) \cup II(SR)$, where $II(RS) \supseteq \Gamma(R, S)$ and $II(SR) \supseteq \Gamma(S, R)$ by Lemma 6. Thus $(S^o)^* \supseteq II(R) [\Gamma(R, S) \cup \Gamma(S, R)]$. Similarly, $(S^o)^* \supseteq ((SR)^o)^* \supseteq II(R) \cup \Gamma(R, S) \cup \Gamma(S, R) \cup R \supseteq II(R)$.

When $S_1$ is a semigroup, this has obvious consequences, first when $S_1 \in B\mathfrak{B}(X)$ and $R = S = S_1$ and second when $S_1$ is in $B\mathfrak{B}(X)$ and $R = S_1^{-1}$ and $S = X - S_1$. These together imply Corollary 5 of [3].

Lemma 7. For any subset $S$ of a topological space $X$ these conditions are equivalent: (1) $S$ is a Baire set, $S \subseteq II(S)$, and $X - S \subseteq X - S$; (2) the equalities (i) $S^o = II(S)^o = (S^o)^o = I(X - S)$ and (ii) $S^* = II(S) = (S^o)^* = (X - S)^*$ are true; (3) equalities (i) and (ii) hold when $S$ and $X - S$ are interchanged.

Taking complements in (i) and (ii) yields (ii) and (i) with $S$ and $X - S$ interchanged; thus (2) and (3) are equivalent. Concerning (1) and (2) we note that by (b) above $S \subseteq II(S)$ is equivalent to (4) $S^* = II(S)$, that $X - S \subseteq X - S$ is equivalent to $(X - S)^* = II(X - S)$, that is, to (5) $S^o = I(X - S)$, and recall that $S$ is a Baire set if and only if (6) $II(S)^o \subseteq I(X - S)$. Obviously (2) now implies (1). Conversely, (1) implies (4), (5), and (6), where (6) implies (7) $II(S) \subseteq I(X - S)^*$ since $II(S) = (II(S)^o)^*$. From (4), (7), and (5) we have $S^* = II(S) \subseteq I(X - S)^* = (S^o)^*$; since $(S^o)^* \supseteq S^*$, (ii) follows.
Taking interiors in (ii) yields \((S^*)^0 = II(S)^0 = (I(X-S)^*)^0\); since 
\[(I(X-S)^*)^0 = ((I - II(X-S))^0)^0 = (X - II(X-S)^0)^0 = X - (II(X-S)^0)^0 = I(X-S)\text{ and (5) holds, (i) now follows also.}\]

Lemmas 6 and 7, which are independent of the other lemmas and corollaries, provide the following mild extension of Zorn's theorem on the structure of a semigroup \(S\) when \(S\) is a Baire set such that \(S\) and \(S^{-1}\) are second category at \(e\) [1, pp. 157–158; 3, Corollary 6].

**Theorem.** Suppose \(S\) is a Baire set and \(S \supset RS\) or \(S \supset SR\) for some \(R\) such that \(II(R)^m \cap II(R^{-1})^n \ni e\) for some \(m\) and \(n \geq 1\). Then (2) and (3) of Lemma 7 are true.

Suppose \(S \supset RS\). From Lemma 6 and the definition of \(\Delta(R, S), S^* \supset II(S) \supset II(RS) \supset \Delta(R, S) \supset II(R)S^*\), so that \(S^* \supset II(S) \supset II(R)S^*\). Multiplying by \(II(R)^k\) we have \(II(R)^k S^* \supset II(R)^{k+1} S^*\), and hence \(S^* \supset II(S) \supset II(R)^k S^*\) for any \(k \geq 1\). When \(e \in II(R)^m\), it is then clear that \(S^* \supset II(S) \supset S^*\), or \(S^* = II(S)\). Since \(S \supset RS\) implies \(X-S \supset R^{-1}(X-S)\), we also have \((X-S)^n = II(X-S)\) in case \(e \in II(R^{-1})^n\) for some \(n \geq 1\). A similar proof applies when \(S \supset SR\). Thus our hypotheses here imply (1) of Lemma 7, and the present conclusion follows.

**References**


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