ON A THEOREM OF RÅDSTRÖM

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The purpose of this note is to give a new and simplified proof of the following theorem.

**Theorem.** Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be an entire function. Denote \( M(r) = \max_{|z|=r} |f(z)| \). Assume that \( \limsup \frac{\log M(r)}{r} = \infty \). Then there exists \( \omega_n, n = 0, 1, \ldots \), with \( |\omega_n| = 1 \), so that the origin is a limit point of the roots of the derivatives of \( k(z) = \sum_{n=0}^{\infty} \omega_n z^n \).

In other words the theorem holds if the order \( \rho \) of \( f(z) \) is greater than 1 or if \( \rho = 1 \) and \( f(z) \) is of maximal type.

This theorem is due to Rådström and was proved by him for the case \( \rho > 1 \) in a recent note. The result as announced here is best possible with respect both to order and to type, as is shown by the example \( e^z \), where \( c \) is a constant (cf. footnote 1, p. 400).

We need the following two lemmas.

**Lemma 1.** Let \( \sum_{n=0}^{\infty} a_n z^n \) be a power series with radius of convergence \( R < \infty \) and such that \( |a_0/a_1| < R \). Then it is possible to find \( w_0, w_1, \ldots \) with \( |w_n| = 1 \) so that \( \sum_{n=0}^{\infty} w_n a_n z^n \) has a zero \( z_0 \) with \( |z_0| \leq |a_0/a_1| \).

**Proof.** We put \( a_0 = 1 \) and \( a_0 + a_1 z = P_1(z) \). Obviously \( P_1(z) \) has a zero with the required property. We proceed by induction. Suppose that we have succeeded in determining \( \omega_0, \omega_1, \ldots, \omega_{n-1} \) such that the polynomial \( P_{n-1}(z) = \sum_{n=0}^{n-1} \omega_n a_n z^n \) has a zero \( z_0 \) with \( |z_0| \leq |a_0/a_1| \). Consider \( P_{n-1}(z) + a_n z^n, |\omega_n| = 1 \). Three cases may occur:

1. The equation \( |P_{n-1}(z)| = |a_n z^n| \) has a solution on \( |z| = |a_0/a_1| \).
2. \( |P_{n-1}(z)| > |a_n z^n| \) for all \( z \) with \( |z| = |a_0/a_1| \).
3. \( |P_{n-1}(z)| < |a_n z^n| \) for all \( z \) with \( |z| = |a_0/a_1| \).

In case 1, it will obviously be possible to choose \( \omega \) so that \( P_{n-1}(z) + a_n z^n = 0 \) on the circle \( |z| = |a_0/a_1| \). In case 2, \( P_{n-1}(z) + a_n z^n \) has by Rouché's theorem as many zeros inside the circle \( |z| = |a_0/a_1| \) as \( P_{n-1}(z) \), that is, at least one, by the induction hypothesis. In case 3, again by Rouché's theorem, \( P_{n-1}(z) + a_n z^n \) has as many zeros in \( |z| = |a_0/a_1| \) as \( P_{n-1}(z) \), that is, \( n \) zeros. In all these cases we can therefore choose \( \omega = \omega_n, |\omega_n| = 1 \) so that \( P_{n-1}(z) + a_n \omega_n z^n \) has a zero in the circle \( |z_0| = |a_0/a_1| \). Consider now the power series \( \sum_{n=0}^{\infty} \omega_n a_n z^n \).

We know that all its partial sums have zeros in or on the circle \( |z| = |a_0/a_1| \). As this circle is strictly inside the circle of convergence

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the same must hold for the infinite series, which proves the lemma.

**Lemma 2.** Let \( \sum_{n=0}^{\infty} a_n z^n \) satisfy the conditions of Lemma 1, and let \( \epsilon > 0 \) be a positive number. Then there exists an integer \( n \) and numbers \( \omega_0, \omega_1, \ldots, \omega_n \) with \( |\omega_0| = 1 \) such that the series \( \sum_{n=0}^{\infty} \omega_n a_n z^n \) has a zero in the circle \( |z| \leq |a_0/a_1| + \epsilon \), irrespective of the choice of the numbers \( \omega_v \) for \( v \geq n+1 \).

**Proof.** Let \( r \) be a number with \( |a_0/a_1| < r < \min (R, |a_0/a_1| + \epsilon) \) and such that the series \( f(z) = \sum_{n=0}^{\infty} \omega_n a_n z^n \) constructed in Lemma 1 has a positive minimum \( m \) on the circle \( |z| = r \). Put \( \delta_0 = \sum_{n=0}^{\infty} \omega_n a_n r^n \).

We have \( \delta_n \to 0 \) monotonically. Choose \( n \) so large that \( 2 \cdot \delta_n < m \), and let \( g(z) \) be any series which coincides with \( f(z) \) in the first \( n + 1 \) terms whereas in the rest of the terms arbitrary changes of the arguments are allowed. Obviously \( |g(z) - f(z)| < 2 \cdot \delta_n \) for \( |z| \leq r \). Therefore, by Rouché's theorem, \( g(z) \) has as many roots in \( |z| \leq r \) as \( f(z) \), that is, at least one (since \( r > |a_0/a_1| \)). This proves the lemma.

In order to prove the theorem we first observe that if \( \lim \sup \log M(r)/r = \infty \), it follows that \( \lim \inf |a_n/(n+1)a_{n+1}| = 0 \), for otherwise there would exist a \( k > 0 \) such that for all sufficiently large \( n \), \( a_{n+1} < ka_n/(n+1) \). Iterating this we would get, for sufficiently large \( n \), \( a_n < ck^n/n! \), which as is well known implies \( \lim \sup \log M(r)/r \leq k \), an evident contradiction. Therefore there exists a sequence \( n_s \) of integers such that \( a_n/(n+1)(a_{n+1}) \to 0 \). We also observe that \( f^{(n)}(z)/n! = a_n + (n+1)a_{n+1}z + \cdots \). Now choose a sequence \( \epsilon_s \) of positive numbers with \( \epsilon_s \to 0 \). According to Lemma 2 we can find numbers \( \omega_{n_1}, \omega_{n_1+1}, \ldots, \omega_{n_1+p_1} \) so that if in \( f^{(n)}(z)/n! \) we multiply each coefficient with the corresponding \( \omega_v \), we shall get a function which has a zero in \( |z| < |a_n/(n+1)a_{n+1}| + \epsilon_1 \), and we shall still be able to choose \( \omega_v \) arbitrarily if \( \mu > n_1 + p_1 \) without destroying this property. Therefore we can repeat this process, now starting with the smallest \( n_s > n_1 + p_1 \). Call that number \( m_2 \) and put \( m_1 = n_1 \). Then we get a new set of \( \omega_v \)'s, \( \omega_{m_1}, \ldots, \omega_{m_2+p_2} \), and if \( \mu > m_2 + p_2 \) we still have the free choice of the \( \omega_v \). Iterating this process we shall obtain a sequence of nonoverlapping blocks of \( \omega \)'s and we complete it if necessary by choosing \( \omega_v \) arbitrarily for those \( v \) which do not correspond to an \( \omega \) in a block. In this way we get a sequence \( \omega_0, \omega_1, \ldots \) and we construct the corresponding power series \( k(z) = \sum_{n=0}^{\infty} \omega_n a_n z^n \). From the construction and Lemma 2 it is then obvious that \( k(z) \) will have the property: \( k^{(n)}(z) \) has a zero \( z_s \), satisfying \( z_s < |a_{m_1}/(m_1+1)a_{m+1}| + \epsilon_s \). As the sequence \( m_1 \) is a subsequence of \( n_s \), and \( \epsilon_s \to 0 \), it is clear that \( z_s \to 0 \), which proves the theorem.

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