The object of the present note is the establishment of the following theorem:

**Theorem 1.** If \( \{ T_n(x) \} \) is a sequence of trigonometric polynomials of order \( n \), and if

\[
(1) \quad f(x) - T_n(x) = O(n^{-\alpha}) \quad (\alpha > 0) \text{ uniformly in } x,
\]

then the conjugate function and the conjugate trigonometric polynomials satisfy

\[
(2) \quad \bar{f}(x) - \bar{T}_n(x) = O(n^{-\alpha} \log n) \quad \text{uniformly in } x.
\]

Furthermore the latter order is in general the best possible.

For the sequence \( \{ T_n(x) \} \) the sequence of partial sums of the Fourier series of \( f(x) \), Salem and Zygmund [3] showed that \( f(x) - s_n(x) = O(n^{-\alpha}) \) for \( \alpha > 0 \) uniformly in \( x \) implied that \( f(x) - s_n(x) = O(n^{-\alpha}) \). Kawata [2] pointed out that for the sequence of Fejer means of the Fourier series of \( f(x) \) and for \( 0 < \alpha < 1 \)

\[
(3) \quad f(x) - \sigma_n(x) = O(n^{-\alpha}) \quad \text{uniformly in } x
\]

implied

\[
(4) \quad \bar{f}(x) - \bar{\sigma}_n(x) = O(n^{-\alpha}) \quad \text{uniformly in } x
\]

while (1) for \( \alpha = 1 \) implied only

\[
(5) \quad \bar{f}(x) - \bar{\sigma}_n(x) = O(n^{-1} \log n) \quad \text{uniformly in } x.
\]

For \( \alpha > 1 \), of course, relation (3) implies that \( f(x) \) is a constant.

Suppose first that \( 0 < \alpha < 1 \). If condition (1) is satisfied, then by a theorem of S. Bernstein [1] \( f(x) \in \text{Lip } \alpha \). Hence, by another result of Bernstein [1], relation (3) holds and thus (4) follows.

The polynomial \( Q_n(x) = T_n(x) - \sigma_n(x) \) has \( |Q_n(x)| \leq Kn^{-\alpha} \) uniformly with respect to \( x \). Hence, there is a constant \( C \) such that

\[
(6) \quad |Q_n(x)| \leq C n^{-\alpha} \log n.
\]

The combination of (4) and (6) gives (2).

The argument for \( \alpha \geq 1 \) proceeds in a similar fashion to that used

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1 Numbers in brackets refer to the references at the end of the paper.
by Salem and Zygmund [3]. Choose \( \beta \) so that \( \alpha = \beta + \epsilon \) and \( 0 < \epsilon < 1 \). If we let \( T_{n-1}(x) = 0 \), \( \Delta_n(x) = T_n(x) - T_{n-1}(x) \), then (1) implies \( f(x) = \sum_{k=0}^{n-1} \Delta_k(x) = O(n^{-\alpha}) \). If we now let \( g(x) = \sum_{k=1}^{n} k^\beta \Delta_k(x) \), we have

\[
g(x) - \sum_{k=1}^{n} k^\beta \Delta_k(x) = \sum_{k=n+1}^{\infty} k^\beta \{ T_{k+1}(x) - T_k(x) \}
\]

\[
= \{ f(x) - T_{n+1}(x) \} \big( n + 1 \big)^\beta 
+ \sum_{k=n+2}^{\infty} \{ k^\beta - (k - 1)^\beta \} \{ f(x) - T_k(x) \}
\]

\[
= O(n^{-\alpha}) \quad \text{uniformly in } x.
\]

Hence by the portion of the theorem already established

\[
g(x) - \sum_{k=1}^{n} k^\beta \Delta_k(x) = O(n^{-\alpha} \log n) \quad \text{uniformly in } x.
\]

Consequently if \( S_n(x) = \sum_{j=n+1}^{\infty} j^\beta \Delta_j(x) \),

\[
\tilde{f}(x) - \tilde{T}_n(x) = \sum_{k=n+1}^{\infty} \Delta_k(x) = \sum_{k=n+1}^{\infty} k^{-\beta} \{ S_k(x) - S_{k+1}(x) \}
\]

\[
= (n + 1)^{-\beta} S_{n+1}(x) + \sum_{k=n+2}^{\infty} \{ k^{-\beta} - (k - 1)^{-\beta} \} S_k(x)
\]

\[
= O(n^{-\alpha}) \quad \text{uniformly in } x.
\]

In order to justify the final remark of the theorem, we note that \( f(x) = 0 \) and the polynomials \( T_n(x) = n^{-\alpha} \sum_{k=1}^{n} k^{-1} \sin kx \) satisfy (1) while at \( x = 0 \), \( \tilde{T}_n(x) \) is of the exact order \( n^{-\alpha} \log n \).

**References**


University of Oregon