GAMES AND SUB-GAMES

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The determination of the value and optimal strategies of a zero-sum, two-person game with a finite number of pure strategies can be a lengthy process, involving, among other things, the calculation of the value

$$\sup_x \inf_y (xB, y),$$

where $B$ is a real matrix with $m$ rows and $n$ columns, $x$ ranges over the set of row vectors with $m$ components, all non-negative and adding up to one, $y$ ranges over the corresponding set of $n$-component column vectors, and the pay-off, $(xB, y)$, indicates the inner product of the two vectors $xB$ and $y$. One device which may simplify a game computation is that of "dominance" or "majorization" [vNM, p. 174] by which the solution of a game is reduced to the solution of a smaller game, that is, one with a smaller number of pure strategies. There is another device which, when conditions are right, may simplify the solution of a game by reducing it to the solution of smaller games. This device, presented here, gives either the value or a bound for it, depending on the information available about the sub-games. It also gives an optimal strategy or a strategy sufficient to insure an outcome not worse than that predicted by the aforementioned bound. It is particularly effective when there are rows (or columns) in $B$, which are constant or have large constant segments.

Let $B$ be a game matrix (rows maximizing) decomposed into

$$B = \{ B_i^j \mid 1 \leq i \leq M, 1 \leq j \leq N \},$$

where $B_i^j$ is a sub-matrix with $m_i$ rows and $n_j$ columns (the $m_i$ rows being independent of $j$ and the $n_j$ columns being independent of $i$). Let the value of $B$ be $v$ and the value of $B_i^j$ be $v_i^j$. Let the set of optimal strategies for the first player in the game be $X = \{ x \}$ and the set of optimal strategies for the first player in the sub-game be $X_i^j = \{ x^i \}$. Let $Y$ and $Y_i^j$ represent the corresponding sets for the

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1 This originally appeared in a RAND report: Total reconnaissance with total countermeasures: Simplified model, August 5, 1949, P-106, Rand Corporation, Santa Monica, California. For the definitions in game theory see [vNM]. See the bibliography at the end of the paper.

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second players. Let $\mathbf{B}$ be the $M \times N$ matrix with entries $v_{ij}$. Let $\bar{x} = \{x_i | 1 \leq i \leq M, x_i \geq 0, \sum x_i = 1\}$ be a typical optimal strategy for the first player in the game with matrix $\mathbf{B}$, $\bar{X}$ the set of optimal strategies for the first player in $\mathbf{B}$, and $\bar{y}$ and $\bar{Y}$ the analogous items for the second player. Let $\bar{v}$ be the value of the game with matrix $\mathbf{B}$.

**Theorem.** $\cap_i X_i^j \neq \Lambda$ for each $i$ implies $v \geq \bar{v}$. If $x^i \in \cap_i X_i^j$ for each $i$ and $\bar{x} \in \bar{X}$, then by playing the vector $\{\bar{x}_1 x^1, \bar{x}_2 x^2, \ldots, \bar{x}_M x^M\}$ (where by this notation we mean the vector each of whose first $m_i$ components are $\bar{x}_i$ multiplied by the appropriate one of the $m_i$ components of $x^1$, and so on) the first player may assure himself of a pay-off of at least $\bar{v}$.

**Proof.** Let a typical strategy for II in game with matrix $B$ be

$$y = \{\beta_1 \bar{y}_1, \ldots, \beta_N \bar{y}_N\}$$

where $\bar{y}_j$ is a vector with $n_j$ non-negative components adding up to one and $\beta_j \geq 0$ for each $j$, $\sum \beta_j = 1$. If I plays

$$\{\bar{x}_1 x^1, \ldots, \bar{x}_M x^M\}$$

then the pay-off

$$\sum_{j=1}^N \left( \sum_{i=1}^M \bar{x}_i x^i B_{ij} \beta_j \bar{y}_j \right) \geq \sum_{j=1}^N \left( \sum_{i=1}^M \bar{x}_i \beta_j \bar{y}_j \right) \geq \bar{v}.$$

**Corollary.** $\cap_j Y_j^i \neq \Lambda$ for each $j$ implies $v \leq \bar{v}$. If $y_j \in \cap_j Y_j^i$ for each $j$ and $\bar{y} \in \bar{Y}$, then by playing the vector $\{\bar{y}_1 y_1, \ldots, \bar{y}_N y_N\}$ the second player may limit his losses to $\bar{v}$.

**Corollary.** $\cap_i X_i^j \neq \Lambda$ for each $i$ and $\cap_j Y_j^i \neq \Lambda$ for each $j$ implies $v = \bar{v}$. $x^i \in \cap_i X_i^j$, $y_j \in \cap_j Y_j^i$, $\bar{x} \in \bar{X}$ and $\bar{y} \in \bar{Y}$ implies $\{x_1 x^1, \ldots, x_M x^M\} \in \bar{X}$ and $\{y_1 y_1, \ldots, y_N y_N\} \in \bar{Y}$.

**Bibliography**


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1 Similar results (unpublished) have been obtained by Gale, Kuhn, and Tucker independently of those of the author. An abstract, apparently motivated by consideration of matrices $B$ which have large constant segments, of these results is D. Gale, H. W. Kuhn, and A. W. Tucker, Bull. Amer. Math. Soc. Abstract 55-11-472.