NUMBER OF INTEGERS IN AN ASSIGNED \( A \), \( P \leq x \) AND PRIME TO PRIMES GREATER THAN \( x^c \)

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In the following:
- \( x \) denotes any real number greater than 1; \( y, c, t \) denote real positive numbers; \( l = \log(x) \);
- \( d, n, m, v, k \) denote integers satisfying \( n \geq 0, d > 0, m > 0, 0 < v \leq k \), \( (v, k) = 1 \);
- \( A(m, v, k) \) denotes the set of integers \( m(v+nk) \) for varying \( n \);
- \( f(m, v, k, x, c) \) denotes the number of integers in \( A(m, v, k) \) less than or equal to \( x \) and prime to primes greater than \( x^c \);
- \( \psi(x, k) \) denotes the number of positive integers less than or equal to \( x \) and prime to \( k \);
- \( F(x, k) = x\psi(k)k^{-1} - \psi(x, k) \); \( b(m, k), B(v, k) \) denote functions, to be defined in the context, of the integral variables involved;
- \( \phi(y) \) and \( q(y) \) denote the functions defined on page 100 of my paper\(^1\) entitled The number of integers \( \leq x \) and free of prime divisors > \( x^c \), and a problem of S. S. Pillai;
- \( \pi(v, k, x) \) denotes the number of primes less than or equal to \( x \) in \( A(1, v, k) \);
- \( P(v, k, x) = \sum_{p \leq x, p = r (mod k)} p^{-1} \); \( p \) denoting a prime number;
- \( \hat{n}(v, k) \) denotes the least positive integer satisfying \( n\hat{n}(v, k) = v \mod k \) for \( (n, k) = 1 \).

[We note that \( |F(x, k)| < \psi(k, k) \).]

Buchstab\(^2\) has proved the result

\[
f(1, v, k, x, c) = x\psi(c)k^{-1} + O(xl^{-1/2}).
\]

The object of this note is to prove the following considerable improvements on this result:

(i) \( f(m, v, k, x, c) = x\phi(c)(mk)^{-1} + b(m, k)xl^{-1}q(c) + O(xl^{-1}) + O(xl^{-2}) \) for \( 1/2 \leq c \leq 1 \),

(ii) \( f(m, v, k, x, c) = x\phi(c)(mk)^{-1} + b(m, k)xl^{-1}q(c) + O(xl^{-3/2}) \) for \( c < 1/2 \) and \( c \geq 1 \), where \( b(m, k) = [m\psi(k, k)]^{-1}\int_1^\infty F(t, k)t^{-2}dt \), and the constant implied in each "O" is dependent only on \( v \) and \( k \).

It will be noted that for \( m = v = k = 1 \), this result reduces to Theorem A' proved in my paper cited above.

The proof follows the lines of that of Theorem A' with appro-

Received by the editors December 24, 1949.


appropriate modifications, of which the less obvious are indicated below:

(1) we replace Lemma 3 by the known results

(a) \( \pi(v, k, x) = [\psi(k, k)]^{-1} \log \log x + O(x^{1/2}) \), "O" depending only on

\( k \),

(b) \( P(v, k, x) = [\psi(k, k)]^{-1} \log \log x + B(v, k) + h(x, v, k)x^{-2} \),

where

\[ h(y, v, k) < (1/24)(1 + 1/2)^2 \text{ for } y \geq \exp(l/2) \text{ and } x > a_{10} > 1. \]

(2) We replace Lemma 6 of Theorem A' by the result (i) above of this theorem. To prove this latter, we observe that on account of the unique factorisation theorem, we have, for \( 1/2 \leq c \leq 1 \),

\[ f(m, v, k, x, 1) - f(m, v, k, x, c) \]

\[ = \sum_{1 \leq n \leq x^{1/2}/m; (n, k) = 1} \left[ \pi \left\{ \frac{n(v, k, k, x)}{mn} \right\} - \pi \left\{ \frac{n(v, k, k, x^c)}{mn} \right\} \right] \]

\[ = \frac{1}{\psi(k, k)} \sum \left[ \text{Li} \left( \frac{x}{mn} \right) - \text{Li}(x^c) \right] + O(x^{1/2}), \]

and

\[ \sum \text{Li} \left( \frac{x}{mn} \right) = \int_{1-0}^{(x^{1-c})/m+0} \frac{x}{mt} \left( \psi(t, k) \right) dt \]

\[ = \psi \left[ \frac{x^{1-c}}{m}, k \right] \text{Li} (x^c) + \frac{x}{m} \int_{1}^{x^{1-c}/m} \frac{\psi(t, k)}{t^2 \log (x/mt)} dt \]

\[ = \psi \left[ \frac{x^{1-c}}{m}, k \right] \text{Li} (x^c) + \frac{x\psi(k, k)}{mk} \int_{1}^{x^{1-c}/m} \frac{dt}{t \log (x/t)} \]

\[ - \frac{x}{m} \int_{1}^{x^{1-c}/m} \frac{F(t, k)}{t^2 \log (x/mt)} dt. \]

Hence it follows that

\[ f(m, v, k, x, c) = \left( \frac{x}{mk} \right) (1 + \log c) + b(m, k) x t^{-1} + O(x^{1/2}) + O(x^{1/2}), \]

for \( 1/2 \leq c \leq 1 \),

which is the desired result, since \( \phi(c) = 1 + \log c \) for \( 1/2 \leq c \leq 1 \), and \( q(1) = 0 \), and \( q(c) = 1 \) for \( 1/2 \leq c < 1 \).

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