NUMBER OF INTEGERS IN AN ASSIGNED \( A, P \leq x \) AND PRIME TO PRIMES GREATER THAN \( x^c \)

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In the following:

\( x \) denotes any real number greater than 1; \( y, c, t \) denote real positive numbers; \( \ell = \log(x) \);

\( d, n, m, v, k \) denote integers satisfying \( n \geq 0, d > 0, m > 0, 0 < v \leq k, (v, k) = 1 \);

\( A(m, v, k) \) denotes the set of integers \( m(v+nk) \) for varying \( n \);

\( f(m, v, k, x, c) \) denotes the number of integers in \( A(m, v, k) \) less than or equal to \( x \) and prime to primes greater than \( x^c \);

\( \psi(x, k) \) denotes the number of positive integers less than or equal to \( x \) and prime to \( k \);

\( F(x, k) = x\psi(k, k)k^{-1} - \psi(x, k) \); \( b(m, k) \), \( B(v, k) \) denote functions, to be defined in the context, of the integral variables involved;

\( \phi(y) \) and \( q(y) \) denote the functions defined on page 100 of my paper\(^1\) entitled The number of integers \( \leq x \) and free of prime divisors \( > x^c \), and a problem of S. S. Pillai;

\( \pi(v, k, x) \) denotes the number of primes less than or equal to \( x \) in \( A(1, v, k) \);

\( P(n, v, k, x) = \sum_{\rho \leq x, \rho = \ell (\text{mod} k)} \rho^{-1}, \rho \) denoting a prime number;

\( n(v, k) \) denotes the least positive integer satisfying \( n(v, k) = v \mod k \) for \( (n, k) = 1 \).

[We note that \( |F(x, k)| < \psi(k, k). \)]

Buchstab\(^2\) has proved the result

\[ f(1, v, k, x, c) = x\phi(c)k^{-1} + O(xl^{-1/2}). \]

The object of this note is to prove the following considerable improvements on this result:

(i) \[ f(m, v, k, x, c) = x\phi(c)(mk)^{-1} + b(m, k)xk^{-1}q(c) + O(x\ell^{-1}) \]

\[ + O(xl^{-2}) \text{ for } 1/2 \leq c \leq 1, \]

(ii) \[ f(m, v, k, x, c) = x\phi(c)(mk)^{-1} + b(m, k)xk^{-1}q(c) + O(x\ell^{-3/2}) \text{ for } \]

\[ c < 1/2 \text{ and } c \geq 1, \text{ where } b(m, k) = [m\psi(k, k)]^{-1}F(l, k)t^{-2}dt, \text{ and the constant implied in each } \]

\[ "O" \text{ is dependent only on } v \text{ and } k. \]

It will be noted that for \( m = v = k = 1 \), this result reduces to Theorem A\(^3\) proved in my paper cited above.

The proof follows the lines of that of Theorem A\(^3\) with appro...
appropriate modifications, of which the less obvious are indicated below:

(1) we replace Lemma 3 by the known results
(a) \( \pi(v, k, x) = [\psi(k, k)]^{-1} \text{Li} (x) + O(x^{l-2}) \), "O" depending only on \( k \),
(b) \( P(v, k, x) = [\psi(k, k)]^{-1} \log \log x + B(v, k) + h(x, v, k)l^{-2} \),
where
\[
h(y, v, k) < (1/24)(1 + 1/2)^2 \quad \text{for} \quad y \geq \exp (l^{1/2}) \quad \text{and} \quad x > a_1 > 1.
\]

(2) We replace Lemma 6 of Theorem A' by the result (i) above of this theorem. To prove this latter, we observe that on account of the unique factorisation theorem, we have, for \( 1/2 \leq c \leq 1 \),

\[
f(m, v, k, x, 1) - f(m, v, k, x, c) = \sum_{1 \leq n \leq x^{l-1}/m} \left[ \pi \left\{ \tilde{n}(v, k), k, \frac{x}{mn} \right\} - \pi \left\{ \tilde{n}(v, k), k, x^c \right\} \right]
\]
\[
= \frac{1}{\psi(k, k)} \sum \left[ \text{Li} \left( \frac{x}{mn} \right) - \text{Li}(x^c) \right] + O(xl^{-2}),
\]
and

\[
\sum \text{Li} \left( \frac{x}{mn} \right) = \int_{1-0}^{(x^{l-1})/m+0} \text{Li} \left( \frac{x}{mt} \right) d\psi(t, k)
\]
\[= \psi \left[ \frac{x^{l-c}}{m}, k \right] \text{Li} (x^c) + \frac{x}{m} \int_{1}^{x^{l-c}/m} \frac{\psi(t, k)}{t^2 \log (x/mt)} dt
\]
\[= \psi \left[ \frac{x^{l-c}}{m}, k \right] \text{Li} (x^c) + \frac{x\psi(k,k)}{mk} \int_{1}^{x^{l-c}/m} \frac{dt}{t \log (x/t)}
\]
\[= \frac{x}{m} \int_{1}^{x^{l-c}/m} \frac{F(t, k)}{t^2 \log (x/mt)} dt.
\]

Hence it follows that

\[
f(m, v, k, x, c) = (x/mk)(1 + \log c) + b(m, k)xl^{-1} + O(x^{l-1}) + O(xl^{-2}),
\]
for \( 1/2 \leq c \leq 1 \),

which is the desired result, since \( \phi(c) = 1 + \log c \) for \( 1/2 \leq c \leq 1 \), and \( g(1) = 0 \), and \( g(c) = 1 \) for \( 1/2 \leq c < 1 \).

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