THE RANGE OF CERTAIN VECTOR INTEGRALS

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1. Introduction. Let $u_1, \ldots, u_n$ be completely additive set functions defined over a Borel field $\mathcal{B}$ of subsets of a space $X$, and let $A$ be any bounded subset of Euclidean $n$-space. With every $\mathcal{B}$-measurable function $f = \alpha(x) = [a_1(x), \ldots, a_n(x)]$ defined on $X$ with range in $A$ we associate the vector $v(f) = (\int a_1(x) du_1, \ldots, \int a_n(x) du_n)$. Our problem is to investigate the range $R$ of the function $v(f)$.

If $A$ consists of the two points $(0, 0, \ldots, 0)$ and $(1, 1, \ldots, 1)$, the functions $f$ have the form $\phi(x), \ldots, \phi(x)$, where $\phi(x)$ is the characteristic function of a $\mathcal{B}$-measurable set $E$, and the range $R$ of $v(f)$ is the range of the vector measure $u_1(E), \ldots, u_n(E)$, $E \in \mathcal{B}$. This case has been treated by Liapounoff who has shown that $R$ is closed and, if $u_1(E), \ldots, u_n(E)$ are nonatomic, convex. A simplified proof of Liapounoff's results has been given by Halmos.

Our results are extensions of those of Liapounoff. We shall show (1) that whenever $A$ is closed, $R$ is closed, and (2) that whenever $u_1, \ldots, u_n$ are nonatomic, $R$ is convex. As will be noted below, result (2) follows directly from the corresponding result of Liapounoff; nevertheless we give an independent proof, as our methods differ in detail, though not in essential idea, from those of Halmos, and the principal tool, Theorem 2, is a curious result of some interest.

We shall sketch here the application of the results to statistical decisions and the theory of games, in the special case where $u_1, \ldots, u_n$ are probability measures. Nature (Player I) chooses an integer $i = 1, \ldots, n$, a point $x$ is then chosen from $X$ according to the distribution $u_i$, and the statistician (Player II) observes $x$. He then choose a point $a = (a_1, \ldots, a_n) \in A$ and loses the amount $a_i$. A strategy for the statistician is a function $f = \alpha(x)$, specifying for each $x \in X$ the point $a$ to be chosen when $x$ is observed, and the vector $v[f]$ is his expected loss vector; its $i$th coordinate is the statistician's expected loss when Nature chooses $i$ and he uses strategy $f$. Thus $R$ is the set of loss vectors the statistician can achieve. Now by a mixed strategy, that is, using $N$ strategies $f_1, \ldots, f_N$ with specified probabilities $\lambda_1, \ldots, \lambda_N$, $\lambda_i \geq 0$, $\sum \lambda_i = 1$, the statistician can achieve

Received by the editors May 3, 1950.


as an expected loss vector precisely the points in the convex set determined by \( R \). Thus whenever \( R \) is convex, any vector which can be obtained with a mixed strategy can already be obtained with a pure strategy; mixed strategies are unnecessary. Closure of \( R \) alone seems to have no particular game theory significance, but if \( R \) is convex and closed, it follows from a theorem of Wald\(^3\) that the statistician has a good pure strategy. Our results, then, have the following consequences: if \( u_1, \ldots, u_n \) are nonatomic, mixed strategies are unnecessary; if in addition \( A \) is closed, the statistician has a good pure strategy.\(^4\)

2. Separation of atomic and nonatomic cases. Let \( u \) be any nonnegative measure such that each \( u_i \) is absolutely continuous with respect to \( u \); there are functions \( p_1(x), \ldots, p_n(x) \) such that 
\[
\int p_i(x)\,du(x) = u_i(E), \quad i = 1, \ldots, n.
\]
Let \( X = X_1 + X_2 \), where \( u \) is nonatomic on \( X_1 \), that is, every set of positive \( u \)-measure contains a set of every smaller positive measure, and completely atomic on \( X_2 \), that is, \( X_2 = S_1 + S_2 + \cdots \), and every \( \mathcal{B} \)-measurable subset \( S \) of \( S_i \) has \( u(S) = 0 \) or \( u(S) = u(S_i) \).\(^5\) Now the range \( R \) of \( v[f] \) is simply the vector sum of \( R_1, R_2 \), the ranges obtained when \( X, u_1, \ldots, u_n \) are contracted to \( X_1, X_2 \). Thus we need consider only the two cases \( u \) nonatomic, \( u \) completely atomic.

3. Atomic case.

**Theorem 1.** If \( u \) is atomic and \( A \) is closed, then \( R \) is closed.

**Proof.** If \( X = S_1 + S_2 + \cdots \) is the decomposition of \( X \) into atoms, every \( \mathcal{B} \)-measurable function is constant almost everywhere over each \( S_i \); we specify the function by the sequence \( \{v_i\} \), where \( v_i \) is its value almost everywhere on \( S_i \). The admissible functions \( f \) are sequences \( \{v_i\} \) of points of \( A \). Then every sequence \( \{f_i\} \) of admissible \( f \)'s has a subsequence converging almost everywhere to a function \( f^* \) (\( A \) is bounded). Since \( A \) is closed, the values of \( f^* \) are in \( A \) so that \( f^* \) is admissible. If \( v(f_i) \to \xi \), then \( v(f^*) = \xi \), so that \( R \) is closed.

4. Nonatomic case: convexity of \( R \).


\(^5\) That such a decomposition exists is well known; see for instance the author's Idempotent Markoff chains, Ann. of Math. vol. 43 (1942) pp. 560–567.
Lemma. If \( u \) is any nonatomic, non-negative measure on a Borel field \( \mathcal{B} \) of subsets of a space \( X \), and \( f(x) \) is any \( u \)-integrable function, there is a set \( S \) with \( u(S) = (1/2)u(X) \), \( \int_S f(x)du = (1/2)f(s) \).

Proof. We merely outline the construction of such a set \( S \); the details are straightforward. Let \( a^* \) be the maximum of all real numbers \( a \) for which \( u\{f \geq a\} \geq (1/2)u(X) \). For every \( a \leq a^* \) there are sets \( S \) with \( u(S) = (1/2)u(X) \) of the form \( U + V + W \), where \( U = \{a < f(x) < b\}, V \) is a subset of \( \{f(x) = a\} \), \( W \) is a subset of \( \{f = b\} \).

Let \( \phi(S) = \int_S f(x)du \). If \( a_1 < a_2 \leq a^* \), for any \( S_1, S_2 \) corresponding to \( a_1, a_2, \phi(S_1) \leq \phi(S_2) \). The minimum value of \( \phi(S) \) occurs with \( a = -\infty \), \( V \) equal to a null set; the maximum value of \( \phi(S) \) occurs at \( a = a^* \), \( b = +\infty \), \( W \) equal to a null set; and \( (1/2)\int f(x)du = a \) lies between these two values. For an \( a \) with \( u\{f = a\} = 0 \), there is only one value of \( \phi(S) \), while if \( u\{f = a\} > 0 \), \( \phi(S) \) increases continuously as \( u(V) \) decreases from \( u\{f = a\} \) to 0, except at \( a = a^* \), where \( u(V) \) may have a positive minimum. If \( \psi_1(a) \), \( \psi_2(a) \) are the minimum and maximum values of \( \phi(S) \) for a given \( a \), \( \psi_1, \psi_2 \) are continuous from the left and right respectively. If \( a_0 \) is the maximum \( a \) with \( \psi_1(a) \leq a \), we have, for \( a > a_0 \), \( \psi_2(a) \geq \psi_1(a) > a \), so that, \( \psi_2(a) \) being continuous from the right, \( \psi_2(a_0) \geq a \). Since for \( S \) corresponding to \( a_0 \) we can make \( \phi(S) \) assume any value between \( \psi_1(a_0), \psi_2(a_0) \), there is an \( S \) corresponding to \( a_0 \) with \( \phi(S) = a \).

Theorem 2. If \( u \) is any nonatomic, non-negative measure on a Borel field \( \mathcal{B} \) of subsets of \( X \) and \( f_1, \ldots, f_n \) are any \( \mathcal{B} \)-measurable functions with \( \int f_i du \) finite, \( i = 1, \ldots, n \), there is a Borel field \( \mathcal{A} \subset \mathcal{B} \) such that \( u \) is nonatomic on \( \mathcal{A} \) and for every \( D \in \mathcal{A} \),

\[
\int_D f_i(x)du = u(D) \int f_i(x)du.
\]

Proof. It is no loss of generality to suppose \( u(X) = 1 \) and use the language of probability. The theorem then asserts that \( E_{\mathcal{A}}(f_i) = E(f_i), i = 1, \ldots, n \), where \( E(f), E_{\mathcal{A}}(f) \) denote the expectation and conditional expectation with respect to the Borel field \( \mathcal{A} \) of the chance variable \( f \).

We first prove the theorem for the single chance variable \( f_1 \). Suppose \( E(f_1) = a \). According to the lemma there is a set \( S \), with \( u(S) = (1/2)u(X) \), \( \int_S f_1 du = (1/2)a \). Applying the lemma to the sets \( S = S_1, CS = S_2, f_1 \), we obtain four disjoint sets \( S_{11}, S_{12}, S_{21}, S_{22} \) such that \( S_{11} + S_{12} = S_1, u(S_{11}) = 1/4, \int_{S_{11}} f_1 du = (1/4)a \). Continuing in this way, and denoting the Borel field determined by the \( 2^n \) sets obtained at the \( n \)th stage by \( \mathcal{B}_n \), we see that \( \mathcal{B}_1 \subset \mathcal{B}_2 \subset \cdots, E_{\mathcal{B}_n}(f_1) = E(f_1), and
\( B_n \) contains \( 2^n \) disjoint sets of measure \((1/2)^n\). If \( \mathcal{A}_1 \) is the smallest Borel field containing \( B_1, B_2, \ldots \), \( u \) is nonatomic on \( \mathcal{A}_1 \) and, according to a theorem of Doob,\(^6\) \( E_{B_n}(f_1) \rightarrow E_{\mathcal{A}_1}(f_1) \). Thus \( E_{\mathcal{A}_1}(f_1) = E(f_1) \).

Now applying the result for a single function to \( E_{\mathcal{A}_1}(f_i) \) with \( B \) replaced by \( \mathcal{A}_1 \), we obtain a Borel field \( \mathcal{A}_2 \subset \mathcal{A}_1 \) on which \( u \) is nonatomic, and on which \( E_{\mathcal{A}_2}[E_{\mathcal{A}_2} f_i] = E_{\mathcal{A}_1}(f_i) \). Thus \( E_{\mathcal{A}_i}(f_i) = E(f_i), i = 1, 2 \).

Continuing in this way, we obtain a decreasing sequence of Borel fields \( \mathcal{A}_1 \supset \mathcal{A}_2 \supset \cdots \supset \mathcal{A}_n \) such that \( \mathcal{A}_n = \mathcal{A} \) has the property asserted in the theorem.

**Theorem 3.** If \( u \) is nonatomic, \( R \) is convex.

**Proof.** Suppose \( v[A_1] = A_1, v[A_2] = A_2 \), where \( f_1 = (a_1(x), \ldots, a_n(x)), f_2 = (a_2(x), \ldots, a_n(x)) \). Let \( \mathcal{A} \) be a Borel field such that \( u \) is nonatomic on \( \mathcal{A} \), and \( E_{\mathcal{A}}(a_i) = E(a_i) \) for \( g = a_1 p_1, \ldots, a_n p_n, a_1 p_2, \ldots, a_n p_2 \). Let \( D \) be a set in \( \mathcal{A} \) with \( u(D) = t, 0 \leq t \leq 1 \), and define \( f = f_1 \) on \( D \), \( f_2 \) on \( CD \). Then

\[
E(f) = \left( \int_D a_1 p_1 du + \int_{CD} a_1 p_1 du, \ldots, \int_D a_n p_n du + \int_{CD} a_n p_n du \right)
\]

As remarked above, Theorem 3 is a direct consequence of the theorem of Liapounoff: we consider the \( 2n \)-dimensional measure \( w(D) = \int_D a_1 p_1 du, \ldots, \int_D a_n p_n du, \int_D a_1 p_2 du, \ldots, \int_D a_n p_2 du \). We have \( w(X) = (r_1, r_2) \), so that for any \( t, 0 \leq t \leq 1 \), by Liapounoff’s theorem, there is a \( D \) with \( w(D) = (tr_1, tr_2) \). Then \( w(CD) = [(1-t)r_1, (1-t)r_2] \) and, defining \( f = f_1 \) on \( D \), \( f = f_2 \) on \( CD \), we obtain \( v(f) = tr_1 + (1-t)r_2 \).

5. **Nonatomic case, \( A \) closed: closure of \( R \).**

**Theorem 4.** Suppose \( A \) is closed, and let \( L_1, \ldots, L_k \) be any linear functions on \( n \)-dimensional space. Let \( \mathcal{R} \) be the closure of \( R \); define \( \lambda_i = \min_{r \in \mathcal{R}} L_i(r) \), \( S_i = \mathcal{R} \{ L_i(r) = \lambda_i \} \), and, inductively for \( 1 < i \leq k \), \( \lambda_i = \min_{r \in \mathcal{R}} L_i(r) \), \( S_i = S_{i-1} \{ L_i(r) = \lambda_i \} \). Then there is a point \( r \in \mathcal{R} \) with \( L_i(r) = \lambda_i, i = 1, \ldots, k \).

**Proof.** We may suppose, choosing additional \( L_i \) if necessary, that there are \( n \) linearly independent linear functions among the \( L_i \). Let \( r^* \) be the point such that \( L_i(r^*) = \lambda_i, i = 1, \ldots, k \). Then \( r^* \in \mathcal{R} \). Let \( \{ r_j \} \) be a sequence of points in \( \mathcal{R} \) with \( r_j \rightarrow r^* \); say \( r_j = v(f_j), f_j = [a_{ij}(x)], \)

\(^6\) Regularity properties of certain families of chance variables, Trans. Amer. Math. Soc. vol. 47 (1940) p. 460, Theorem 1.3.
\[ \ldots , a_{n_j}(x) \]. Define \( \phi_{ij}(x) = L_i \left[ a_{ij}(x) p_1(x), \ldots, a_{nj}(x) p_n(x) \right] \). Then \( \int \phi_{ij}(x) \, du \rightarrow \lambda_i \) as \( j \rightarrow \infty \). We shall show that there is a subsequence \( j_i \) such that \( \phi_{ij_i}(x) \rightarrow \phi(x) \) as \( t \rightarrow \infty \) except on an \( \varepsilon \)-set of \( \mu \)-measure zero.

Define
\[
\phi_N(x) = \min \left[ \phi_{11}(x), \ldots, \phi_{1N}(x) \right],
\]
\[
S_{kn} = \{ \phi_{ij} > \phi_N \text{ for } j < k, \phi_{ik} = \phi_N \},
\]
\[
f_N^* = f_{k} \text{ on } S_{kn}.
\]

Then \( L_i(f_N^*) = \int \phi_N(x) \, du \). Now \( \phi_N(x) \leq \phi_{1N}(x) \), so that \( L_i(f_{k\infty}^*) \leq L_i(f_N^*) \), \( L_i(f_{k\infty}^*) \rightarrow \lambda_i \). Let \( \phi(x) = \lim \phi_N(x) \), \( N \rightarrow \infty \). Then \( \phi_{1N} \geq \phi \) for all \( x \) and \( \int \phi_{1N} \, du \rightarrow \int \phi \, du \). Consequently \( \phi_{1N} \rightarrow \phi \) in \( u \)-measure, and there is a subsequence of \( \phi_{1N} \rightarrow \phi \) almost everywhere. Suppose we have found a subsequence of \( j \) for which \( \phi_{ij} \) converges almost everywhere to a function \( \phi_i(x) \) for \( 1 \leq i < m \). To simplify notation, suppose the original sequence \( \phi_{ij} \) has this property. If \( \phi_{mj} \) does not converge in \( \mu \)-measure, there is an \( \varepsilon > 0 \) and a sequence of integers \( s_1, t_1, s_2, t_2, \ldots, s_j, t_j \) becoming infinite with \( j \), for which the set \( T_j = \{ \phi_{mj} = \phi_{mt} \} \) has \( u(T_j) > \varepsilon \). Define \( g_j = f_{mj} \) on \( T_j \), \( g_j = f_{mt} \) on \( C(T_j) \). Then \( L_i(g_j) \rightarrow \lambda_i \) for \( i < m \), and \( L_m(g_j) \leq L_m(f_{sj}) - \varepsilon^2 \), so that \( \lim \sup L_m(g_j) \leq \lambda_m - \varepsilon^2 \). Since \( v_j = v(g_j) \subset R \), there is a subsequence of \( v_j \) approaching \( v_0 \subset R \). We have \( L_i(v_0) = \lambda_i \) for \( i < m \), \( L_m(v_0) \leq \lambda_m - \varepsilon^2 \), contradicting the definition of \( \lambda_m \). Hence \( \phi_{mj} \) converges in \( \mu \)-measure to a limiting function \( \phi_m(x) \) and there is a subsequence converging almost everywhere to \( \phi_m(x) \). This completes the induction.

We may now suppose, replacing \( f \) by an appropriate subsequence, that \( \phi_{ij}(x) \rightarrow \phi_i(x) \) almost everywhere as \( j \rightarrow \infty \) for \( i = 1, \ldots, k \); that is, \( L_i \left[ a_{ij} p_1, \ldots, a_{nj} p_n \right] \rightarrow \phi_i(x) \), \( i = 1, \ldots, k \). Since \( L_1, \ldots, L_k \) contain \( n \) linearly independent functions, the sequence of points \( \{ (a_{ij}, a_{1j} p_1, \ldots, a_{nj} p_n) \} \) converges almost everywhere to a limiting point \( w(x) = (w_1(x), \ldots, w_n(x)) \). If \( p_1, \ldots, p_n \) never vanish, this implies \( a_{ij}(x) \) converges to a limiting function \( a^*(x) \) almost everywhere; since \( A \) is closed, the values of \( a^*(x) \) are in \( A \), \( f = a^*(x) \) is an admissible function, and \( L_i(f) = \lim L_i(f_j) = \lambda_i, i = 1, \ldots, n \), and the proof is complete. If \( p_1, \ldots, p_n \) sometimes vanish, we make the following modification. We need only find an \( a^*(x) = (a_1^*(x), \ldots, a_n^*(x)) \) such that the \( t \)th component \( a_{ij}^*(x) \) of \( a_{ij}(x) \) converges to \( a_t^*(x) \) for all \( x \) with \( p_t(x) \neq 0 \); the values of \( a_t^*(x) \) on \( \{ p_t = 0 \} \) do not influence \( v[a^*] \). For any subset \( \alpha = (i_1, \ldots, i_\ell) \) of \( (1, \ldots, n) \), let \( U_\alpha \) be the \( \varepsilon \)-set where \( p_t \neq 0 \) for \( t \in \alpha, p_t = 0 \) for \( t \notin \alpha \). Let \( h_\alpha \) be a Baire func-

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\footnote{For brevity we write \( L_i(f) \) for \( L_i[v(f)] \).}
tion mapping $c$-dimensional space into $n$-dimensional space in such a way that every point $(x_1, \cdots, x_c)$ in the projection of $A$ on the $c$-dimensional subspace $(a_{1i}, \cdots, a_{ci})$, that is, every point $(x_1, \cdots, x_c)$ for which there is a point $a = (a_1, \cdots, a_n) \in A$ with $a_{1i} = x_1, \cdots, a_{ci} = x_c$, has $h_a(x_1, \cdots, x_c) = a^* = (a_{1i}^*, \cdots, a_{ci}^*) \in A$, with $a_{1i}^* = x_1, \cdots, a_{ci}^* = x_c$. On $U_a$, $a_{ij}(x)$ converges for $t \in (a)$, say to $a_{ij}^*(x)$. Then $a^*(x) = h_a[a_{1i}^*(x), \cdots, a_{ci}^*(x)]$ on $U_a$ is the required function.

**Theorem 5.** If $u$ is nonatomic and $A$ is closed, then $R$ is closed.

**Proof.** According to Theorem 3, $R$ is convex. With the property of $R$ proved in Theorem 4, its closure follows from the following fact about convex sets.

**Lemma.** Let $B$ be any closed bounded convex set, and let $D$ be any convex set with the property (*): For any linear functions $L_1, \cdots, L_k$, if we define $\lambda_1 = \min_{x \in B} L_1(x)$, $B_1 = B\{L_1(x) = \lambda_1\}$ and, inductively for $1 < i \leq k$, $\lambda_i = \min_{x \in B_{i-1}} L_i(x)$, $B_i = B_{i-1}\{L_i(x) = \lambda_i\}$, there is a point $d \in BD$ with $L_i(d) = \lambda_i$, $i = 1, \cdots, k$. Then $D$ contains $B$.

This is easily established by induction on the dimension of $B$. If $B$ is one-dimensional, that is, a closed interval, $D$ must contain the end points of $B$ so that $D \supset B$. Suppose the lemma established for sets $B$ of dimension less than $s$, and let $B$ be $s$-dimensional. The intersection $B_1$ of $B$ with any supporting hyperplane $L_1(x) = \lambda_1$, where $\lambda_1 = \min_{x \in B} L_1(x)$, is a closed convex set of dimension less than $s$ and has the property (*) relative to $D$. By the induction hypothesis, $D \supset B_1$. Thus $D$ contains the intersection of $B$ with every supporting hyperplane, so that $D \supset B$.

**Theorem 6.** If $A$ is closed, $R$ is closed.

**Proof.** This follows from Theorems 1 and 5, by splitting $u$ into atomic and nonatomic parts with ranges $R_1, R_2$ and noting that $R$ is the vector sum of the closed sets $R_1, R_2$.

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8 Such a Baire function may be constructed as follows: To simplify notation, say $a = (1, \cdots, c)$. Let $A_d$ be the projection of $A$ on its first $d$ coordinates, $c \leq d \leq n$. Then each $A_d$ is closed and is the projection of $A_{d+1}$ on its first $d$ coordinates. If we can construct a Baire function $h_d$ mapping $d$-space into $(d+1)$-space with the required property, where $A$ is replaced by $A_{d+1}$, the Baire function $h_{n-1}[h_{n-2}[\cdots [h_d(x_1, \cdots, x_c) \cdots]]$ is the required function. Thus we may suppose $c = n-1$. The function $\psi(x_1, \cdots, x_{n-1}) = \min_{(a_1, \cdots, a_{n-1}, \psi) \in A_n}$ on $A_{n-1}$, and $0$ elsewhere, is lower semi-continuous on the closed set $A_{n-1}$ and so is a Baire function, and the function $h(x_1, \cdots, x_{n-1}) = [x_1, \cdots, x_{n-1}, \psi(x_1, \cdots, x_{n-1})]$ is the required function.