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**ON THE CONTINUITY OF PARAMETRIC LINEAR OPERATIONS**

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The proofs of the theorems asserting strong continuity for semi-groups of linear operations in Banach spaces usually involve measurability and integrability of Banach-space-valued functions [2, pp. 183–184]. A theorem of this type in which the assumptions and the proof are purely topological is given below.

Let $G$ be both a topological space and an additive group, and let $H$ be a subset of $G$. For each $h$ in $H$ let $D(h)$ be all points $g$ in $H$ satisfying these two conditions: (1) $h - g \in H$, (2) for each open set $N_g$ about $g$ there is an open set $N_h$ about $h$ such that $h - g + (H \cap N_g) \supset H \cap N_h$. Letting $X$ be a complex linear normed space, a set $\Gamma = [y]$ of bounded complex linear functionals on $X$ is a total set for $E$, $E$ a subset of $X$, if $\|x\| = \sup |\gamma(x)|, \gamma \in \Gamma$ holds for every $x$ in the smallest linear subspace containing $E$. A function $T_h$ on $H$ to the

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2 Numbers in brackets refer to the references at the end of the paper.
space $B(X)$ of bounded linear operations on $X$ to $X$ is additive if $T_{h+k} = T_h(T_k)$ whenever $h$, $k$, and $h+k$ are in $H$. Finally, we define a function $\phi(a)$ on a topological space $A$ to a metric space $B$ to be residually separably-valued if the set $\phi(A') = \{\phi(a) | a \in A'\}$ is separable in $B$ for some set $A'$ residual in $A$, that is, for some $A'$ having its complement of the first category.

**Theorem.** Let $H$ be a subset of $G$ and let $T_h$ be additive on $H$ to $B(X)$. Suppose that (i) $D(h)$ is second category for each $h \in H$, and that (ii) for each $x_0$ in $X$ the function $\phi(h) = T_h(x_0)$ on $H$ to $X$ is residually separably-valued and there is a total set $\Gamma(x_0)$ for $\phi(H)$ such that $\gamma(\phi(h))$ is continuous on $H$ to complex numbers for each $\gamma \in \Gamma(x_0)$. Then $\phi(h)$ is continuous on $H$ to $X$ for each $x_0 \in X$.

Fixing $x_0$ in $X$, the idea of the proof is to show, using (ii), that the set $C$ of points of continuity of $\phi$ is residual, and then to apply (i) to show that $C = H$.

The proof that $C$ is residual is due to Alexiewicz and Orlicz [1, pp. 107–108 and 114–115] in a slightly more restricted case. Their arguments can be adapted here as follows. We first observe that if $S$ is any closed sphere in $X$ with center $y$ in $\phi(H)$ and radius $r$, then $\phi^{-1}(S)$ is closed in $H$; for since $\Gamma(x_0)$ is a total set for $\phi(H)$ and $\gamma(\phi(h))$ is continuous for $\gamma \in \Gamma(x_0)$, clearly $\phi^{-1}(S) = H[h \in H | \gamma(\phi(h)) \leq r] = H[h \in H | \gamma(\phi(h)) \leq r]$, a closed set. To establish that $C$ is residual it is enough to prove this: if $\phi$ is any function on a topological space $H$ to a metric space $X$ and $\phi$ is residually separably-valued and $\phi^{-1}(S)$ is closed whenever $S$ is a closed sphere with center in $\phi(H)$, then $\phi$ has its points of continuity forming a residual set $C$. Let $R$ be a residual set such that $\phi(R)$ is separable in $X$. Since $\phi(R)$ is separable, there exist countably many closed spheres $\{S_n\}$ with centers in $\phi(R)$ such that for any open set $V$ in $X$ we have $V \cap \phi(R) = \bigcup [S_n \cap \phi(R)] = \bigcup [S_n \cap \phi(R)]$. Let $R'$ be the complement of $R$ in $H$ and set $P_V = R' \cap \phi^{-1}(V)$. Then $\phi^{-1}(V) = P_V \cup [R \cap \phi^{-1}(V)] \subseteq P_V \cup \phi^{-1}(V) \subseteq P_V \cup (\bigcup (S_n)) \subseteq \phi^{-1}(V)$, since each $S_n \subseteq V$; hence $\phi^{-1}(V) = P_V \cup (F_n)$, where $F_n = \phi^{-1}(S_n)$, and in particular $F_n \subseteq \phi^{-1}(V)$, where $E^0$ denotes the interior of any set $E$. Now set $Q = R' \cap (\bigcup (S_n))$. Thus $Q$ is a first category set. Moreover, clearly $\phi^{-1}(V) \subseteq P_V \cup (\bigcup (S_n)) \subseteq Q \cup \phi^{-1}(V)$, so $Q$ is residual and each $F_n$ is closed, $Q$ is a first category set. Moreover, clearly $\phi^{-1}(V) \subseteq P_V \cup (\bigcup (S_n)) \subseteq Q \cup \phi^{-1}(V)$, so $Q$ is a first category set. Moreover, clearly $\phi^{-1}(V) \subseteq P_V \cup (\bigcup (S_n)) \subseteq Q \cup \phi^{-1}(V)$. Now in $H$ consider any point $g$ not in $Q$ and any open $V$ containing $\phi(g)$; since $g \in \phi^{-1}(V) \subseteq Q \cup \phi^{-1}(V)$, and $g \in Q$, obviously $g \in \phi^{-1}(V)$, and hence $\phi$ is continuous at $g$. Thus $C$ is residual.
Let $h$ be fixed in $H$. Since $D(h)$ is second category there exists a point $g$ in $D(h)$ at which $\phi$ is continuous. Set $\rho = \|T_{h-g}\|$ and let $\epsilon > 0$ be given. Having $\phi$ continuous at $g$, there is an open set $N_g$ about $g$ such that $\|\phi(k) - \phi(g)\| < \epsilon/\rho$ whenever $k \in H \cap N_g$. Then, since $g \in D(H)$, there is an open set $N_h$ about $h$ such that $h - g + (H \cap N_g) \supset H \cap N_h$.

Consider any $h'$ in $H \cap N_h$. Writing $h' = h - g + k$ where $k \in H \cap N_g$, we have $\|\phi(h') - \phi(h)\| = \|\phi(h - g + k) - \phi(h - g + g)\| = \|T_{h-g}(T_k(x_0)) - T_{h-g}(T_{h-g}(x_0))\| \leq \rho\|\phi(k) - \phi(g)\| < \epsilon$, showing that $\phi$ is continuous at $h$ and ending the proof.

It is easy to verify that assumption (i) in the theorem is satisfied when (i') the group sum $h + k$ in $G$ is continuous in $k$ and for each $h$ in $H$ the set $H \cap (-H + h)$ is in the interior of $H$ and is second category. Statement (i') in turn is implied by this; (i'') $h + k$ is continuous in each variable in $G$, $G$ is second category, $H$ and $-H$ are open, and $H \subseteq H + H$. Condition (i'') holds, for example, when $G$ is a second category linear topological space and $H$ is an open convex set having the zero element as a limit point. Assumption (ii) of the theorem follows if $H$ contains a countable dense subset and $\gamma(T_h(x_0))$ is continuous on $H$ to complex numbers for each $x_0$ in $X$ and each bounded linear functional $\gamma$ on $X$. In particular, Theorem 9.2.2 of [2] now results.

References


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