ON THE CONTINUITY OF PARAMETRIC LINEAR OPERATIONS

B. J. PETTIS

The proofs of the theorems asserting strong continuity for semi-groups of linear operations in Banach spaces usually involve measurability and integrability of Banach-space-valued functions [2, pp. 183–184]. A theorem of this type in which the assumptions and the proof are purely topological is given below.

Let $G$ be both a topological space and an additive group, and let $\mathcal{H}$ be a subset of $G$. For each $h$ in $\mathcal{H}$ let $D(h)$ be all points $g$ in $\mathcal{H}$ satisfying these two conditions: (1) $h - g \in \mathcal{H}$, (2) for each open set $N_g$ about $g$ there is an open set $N_h$ about $h$ such that $h - g + (H \cap N_h) \supset H \cap N_h$. Letting $X$ be a complex linear normed space, a set $\Gamma = \{\gamma\}$ of bounded complex linear functionals on $X$ is a total set for $\mathcal{E}$, $\mathcal{E}$ a subset of $X$, if $\|x\| = \sup \{ |\gamma(x)|, \gamma \in \Gamma \}$ holds for every $x$ in the smallest linear subspace containing $E$. A function $T_h$ on $H$ to the

1 This paper was written under Contract N7-onr-434, Task Order III, Navy Department (The Office of Naval Research).

2 Numbers in brackets refer to the references at the end of the paper.
space $B(X)$ of bounded linear operations on $X$ to $X$ is additive if $T_{h+k} = T_h(T_k)$ whenever $h$, $k$, and $h+k$ are in $H$. Finally, we define a function $\phi(a)$ on a topological space $A$ to a metric space $B$ to be residually separably-valued if the set $\{\phi(a) \mid a \in A'\}$ is separable in $B$ for some set $A'$ residual in $A$, that is, for some $A'$ having its complement of the first category.

**Theorem.** Let $H$ be a subset of $G$ and let $T_h$ be additive on $H$ to $B(X)$. Suppose that (i) $D(h)$ is second category for each $h \in H$, and that (ii) for each $x_0$ in $X$ the function $\phi(h) = T_h(x_0)$ on $H$ to $X$ is residually separably-valued and there is a total set $\Gamma(x_0)$ for $\phi(H)$ such that $\gamma(\phi(h))$ is continuous on $H$ to complex numbers for each $\gamma \in \Gamma(x_0)$. Then $\phi(h)$ is continuous on $H$ to $X$ for each $x_0 \in X$.

Fixing $x_0$ in $X$, the idea of the proof is to show, using (ii), that the set $C$ of points of continuity of $\phi$ is residual, and then to apply (i) to show that $C = H$.

The proof that $C$ is residual is due to Alexiewicz and Orlicz [1, pp. 107–108 and 114–115] in a slightly more restricted case. Their arguments can be adapted here as follows. We first observe that if $S$ is any closed sphere in $X$ with center $y$ in $\phi(H)$ and radius $r$, then $\phi^{-1}(S)$ is closed in $H$; for since $\Gamma(x_0)$ is a total set for $\phi(H)$ and $\gamma(\phi(h))$ is continuous for $\gamma \in \Gamma(x_0)$, clearly $\phi^{-1}(S) = H \{ h \mid \gamma(\phi(h)) - y \leq r \}$, a closed set. To establish that $C$ is residual it is enough to prove this: if $\phi$ is any function on a topological space $H$ to a metric space $X$ and $\phi$ is residually separably-valued and $\phi^{-1}(S)$ is closed whenever $S$ is a closed sphere with center in $\phi(H)$, then $\phi$ has its points of continuity forming a residual set $C$. Let $R$ be a residual set such that $\phi(R)$ is separable in $X$. Since $\phi(R)$ is separable, there exist countably many closed spheres $S_n$ with centers in $\phi(R)$ such that for any open set $V$ in $X$ we have $V \cap \phi(R) = \bigcup S_n \cap \phi(R)$, where $n$ ranges over all $n$ such that $S_n \subset V$. Let $R'$ be the complement of $R$ in $H$ and set $P_V = R' \cap \phi^{-1}(V)$. Then $\phi^{-1}(V) = P_V \cup \{ R \cap \phi^{-1}(V) \} \subset P_V \cup \phi^{-1}(\Gamma(\phi(R))) \subset P_V \cup (\cup S_n) \subset \phi^{-1}(V)$, where $\gamma \in \Gamma(x_0)$, and in particular $P_V \cap \phi^{-1}(S_n) \subset \phi^{-1}(V)$, where $E^0$ denotes the interior of any set $E$. Now set $Q = R' \cap (\cup S_n \cap \phi(R))$. Since $R$ is residual and each $S_n$ is closed, $Q$ is a first category set. Moreover, clearly $\phi^{-1}(V) \subset P_V \cup (\cup S_n) \subset Q \cup \phi^{-1}(V)$. Now in $H$ consider any point $g$ not in $Q$ and any open $V$ containing $\phi(g)$; since $g \in \phi^{-1}(V) \subset Q \cup \phi^{-1}(V)$, and $g \in Q$, obviously $g \in \phi^{-1}(V)$ and hence $\phi$ is continuous at $g$. Thus $C$ is residual.
Let \( h \) be fixed in \( H \). Since \( D(h) \) is second category there exists a point \( g \) in \( D(h) \) at which \( \phi \) is continuous. Set \( \rho = \| T_{h-g} \| \) and let \( \epsilon > 0 \) be given. Having \( \phi \) continuous at \( g \), there is an open set \( N_g \) about \( g \) such that \( \| \phi(k) - \phi(g) \| < \epsilon / \rho \) whenever \( k \in H \cap N_g \). Then, since \( g \in D(H) \), there is an open set \( N_h \) about \( h \) such that \( h - g + (H \cap N_g) \supset H \cap N_h \). Consider any \( h' \) in \( H \cap N_h \). Writing \( h' = h - g + k \) where \( k \in H \cap N_g \), we have
\[
\| \phi(h') - \phi(h) \| = \| \phi(h - g + k) - \phi(h - g + g) \| = \| T_{h-g}(T_k(x_0)) - T_{h-g}(T_k(x_0)) \| \leq \rho \| \phi(k) - \phi(g) \| < \epsilon,
\]
showing that \( \phi \) is continuous at \( h \) and ending the proof.

It is easy to verify that assumption (i) in the theorem is satisfied when (i') the group sum \( h + k \) in \( G \) is continuous in \( k \) and for each \( h \) in \( H \) the set \( H \cap (H + H) \) is in the interior of \( H \) and is second category. Statement (i') in turn is implied by this; (i'') \( h + k \) is continuous in each variable in \( G \), \( G \) is second category, \( H \) and \( -H \) are open, and \( H \subset H + H \). Condition (i'') holds, for example, when \( G \) is a second category linear topological space and \( H \) is an open convex set having the zero element as a limit point. Assumption (ii) of the theorem follows if \( H \) contains a countable dense subset and \( \gamma(T_h(x_0)) \) is continuous on \( H \) to complex numbers for each \( x_0 \) in \( X \) and each bounded linear functional \( \gamma \) on \( X \). In particular, Theorem 9.2.2 of [2] now results.

**References**


Tulane University and Princeton University