

AN ERGODIC THEOREM FOR A NONCOMMUTATIVE SEMIGROUP OF LINEAR OPERATORS

J. E. L. PECK

Introduction. The question of ergodicity of a semigroup of bounded linear operators on a Banach space has been reduced, by Alaoglu and Birkhoff [1],¹ Day [2, 3], and Eberlein [4], to the study firstly of the ergodicity of the semigroup itself and secondly, of the ergodicity of each element of the Banach space with respect to this ergodic semigroup. In the case of a bounded and commutative semigroup operating on a reflexive Banach space, the ergodicity has been established [2, Corollary 6], [4, Corollary 5.1, Theorem 5.5]. For noncommutative semigroups, Alaoglu and Birkhoff have proved ergodicity by restricting the Banach space [1, Theorem 6]. A general discussion of the noncommutative case is given by Day [3].

The following is an attempt to prove ergodicity, in the strong topology, for a certain noncommutative case, where we use a weakened form of the commutative law. Our method uses a fixed point theorem of Kakutani [5], and a theorem about the group of cluster points of a directed subsemigroup, which we have developed elsewhere [6].

It should be observed, however, that the theorem which is proved here, though being similar to, is not a generalization of the results mentioned above. In fact it is apparently independent of them, because we postulate here a compactness in the strong topology of operators; whereas in the analogous theorems of Day and Eberlein, a kind of weak compactness is used, by considering bounded semigroups acting on reflexive spaces, a natural legacy from the study of Hilbert space. However our theorem is a generalization of a theorem of Yosida [7, Theorem 2].

Preliminary theorems. In this section we prove a theorem in a slightly more general form than we shall require. We need the following notions. A linear space over the field of complex numbers is a *linear topological space* [8] if it has a separated topology [13], which is such that all operations are continuous. More precisely, there is a fundamental system of neighborhoods of the zero element $\{V_\alpha | \alpha \in \Delta\}$, with the following properties:

- (1) The intersection of all V_α is the zero element.

Presented to the Society, April 29, 1950; received by the editors June 22, 1950.

¹ Numbers in brackets refer to the references at the end of the paper.

- (2) Given $\alpha, \beta \in \Delta$ we may find $\gamma \in \Delta$ such that $V_\gamma \subseteq V_\alpha \cap V_\beta$.
- (3) For any $\alpha \in \Delta$ there is a $\beta \in \Delta$ with $V_\beta + V_\beta \subseteq V_\alpha$.
- (4) For any $\alpha \in \Delta$ there is a $\beta \in \Delta$ such that if c is a complex number and $|c| \leq 1$, then $cV_\beta \subseteq V_\alpha$.
- (5) For any element s of the space and any $\alpha \in \Delta$ there is a positive real number r such that $s \in rV_\alpha$.

Also, this system is *locally convex* if in addition

- (6) For any $\alpha \in \Delta$ there is a $\beta \in \Delta$ such that $V_\beta \subseteq V_\alpha$ and V_β is convex.

A subset \mathcal{S} of a linear topological space \mathcal{L} is called *bounded* if to each $\alpha \in \Delta$ there is a complex scalar c such that $\mathcal{S} \subseteq cV_\alpha$, and it is called *totally bounded* if to each $\alpha \in \Delta$ there is a finite set $s_1, \dots, s_n \in \mathcal{L}$ such that $\mathcal{S} \subseteq \bigcup_{i=1}^n [s_i + V_\alpha]$. We observe that we may restrict the finite set to lie in \mathcal{S} [10]. A linear topological space is called *boundedly complete* [9] if every bounded, fundamental directed set [11] is convergent. Finally a linear transformation ϕ of a linear topological space to itself is called *congruent* [5] if $\phi V_\alpha = V_\alpha$ for all $\alpha \in \Delta$.

We shall use a fixed point theorem of Kakutani [5] which appears in the literature without proof. With the permission of the author we reproduce the proof here, for the sake of completeness. This proof makes use of Maak's Lemma [12] which we quote here.

THEOREM 1 (MAAK). *If the set M is partitioned into n subsets in two different ways, $M = A_1 \cup \dots \cup A_n = B_1 \cup \dots \cup B_n$, so that any r of the A_i 's contain at most r of the B_j 's, then there is a common representative system. That is, the subscripts may be reordered in such a way that $A_i \cap B_i \neq \emptyset, i = 1, \dots, n$.*

THEOREM 2 (KAKUTANI). *Let \mathcal{L} be a locally convex linear topological space. Let \mathcal{B} be a compact, convex subset of \mathcal{L} , and let Φ be any set of linear (more generally affine) congruent transformations of \mathcal{L} into itself, each of which takes \mathcal{B} onto itself. Then there is an element $p \in \mathcal{B}$ which is such that $\phi p = p$ for all $\phi \in \Phi$.*

PROOF. We may assume that all $\{V_\alpha | \alpha \in \Delta\}$ are convex neighborhoods and that $V_\alpha = -V_\alpha$. Since \mathcal{B} is compact, then for each $\alpha \in \Delta$ we may find a finite set $b_1, \dots, b_n \in \mathcal{B}$ such that \mathcal{B} is contained in $V_\alpha(b_1) \cup \dots \cup V_\alpha(b_n)$, where $V_\alpha(b) = V_\alpha + b$. We may suppose also that n is chosen so that it is minimal. Then because ϕ is congruent, we have

$$\mathcal{B} = \phi \mathcal{B} \subseteq \phi V_\alpha(b_1) \cup \dots \cup \phi V_\alpha(b_n) = V_\alpha(\phi b_1) \cup \dots \cup V_\alpha(\phi b_n).$$

Put

$$A_1 = V_\alpha(b_1) \cap \mathcal{B}, \quad A_k = [V_\alpha(b_k) - \bigcup_{i=1}^{k-1} A_i] \cap \mathcal{B}, \quad k = 2, \dots, n,$$

$$B_1 = V_\alpha(\phi b_1) \cap \mathcal{B}, \quad B_k = [V_\alpha(\phi b_k) - \bigcup_{i=1}^{k-1} B_i] \cap \mathcal{B}, \quad k = 2, \dots, n.$$

Then because of the choice of n , none of these sets is vacuous. We thus have two partitions $A_1 \cup \dots \cup A_n = B_1 \cup \dots \cup B_n = \mathcal{B}$ which satisfy the postulates of Theorem 1. In fact, if there are r of the A 's which contain $r+1$ of the B 's, then replacing these B 's by the A 's which contain them, we are led to a covering of \mathcal{B} with $n-1$ elements, contradicting the choice of n . Thus using the conclusions of Theorem 1, it is possible to reorder the set b_1, \dots, b_n , say to c_1, \dots, c_n , such that $V_\alpha(b_i) \cap V_\alpha(\phi c_i) \neq \emptyset, i = 1, \dots, n$.

Put

$$a_\alpha = [b_1 + \dots + b_n]/n = [c_1 + \dots + c_n]/n.$$

Then

$$\phi a_\alpha = [\phi b_1 + \dots + \phi b_n]/n = [\phi c_1 + \dots + \phi c_n]/n,$$

$$a_\alpha - \phi a_\alpha = [(b_1 - \phi c_1) + \dots + (b_n - \phi c_n)]/n$$

$$\in [2V_\alpha + \dots + 2V_\alpha]/n \subseteq 2V_\alpha.$$

Thus to each α we associate a point a_α independently of ϕ , which is moved a small distance by ϕ . This directed set $\{a_\alpha | \alpha \in \Delta\}$, being in a compact space, will have at least one cluster point [14]. Call it p . We prove that $\phi p = p$. In fact, given $\beta \in \Delta$, we may find $\alpha > \beta$ such that $a_\alpha \in V_\beta(p)$, whence $\phi a_\alpha \in V_\beta(\phi p)$. Thus

$$\phi p - p = \phi p - \phi a_\alpha + \phi a_\alpha - a_\alpha + a_\alpha - p$$

$$\in V_\beta + 2V_\beta + V_\beta \subseteq 4V_\beta.$$

But since β was arbitrary, this proves that $\phi p = p$.

THEOREM 3. *Let \mathcal{E} be a boundedly complete, locally convex, linear topological space, which is also an abstract algebra, where the new operation satisfies the conditions*

- (1) *if \mathcal{A} is a totally bounded subset of \mathcal{E} , then given $\alpha \in \Delta$ we may find $\beta \in \Delta$ such that $\mathcal{A}V_\beta \subseteq V_\alpha$, and*
- (2) *If A is an element of \mathcal{E} then given $\alpha \in \Delta$ we may find $\beta \in \Delta$ such that $V_\beta A \subseteq V_\alpha$.*

Let \mathcal{G} be a totally bounded, multiplicative group in \mathcal{E} , then there is an element $p \in \text{Cl}(\text{con } \mathcal{G})$ (the closed convex set generated by \mathcal{G}) which is such that $gp = p = pg$ for all $g \in \mathcal{G}$, and $p^2 = p$.

PROOF. Let \mathcal{L} be the set of all bounded functions defined on \mathcal{G} with values in \mathcal{E} . Then \mathcal{L} is a linear space in the usual way [10, Defi-

nition 11]. In \mathcal{L} we introduce the uniform function topology, generated by the topology of \mathcal{E} , that is, $\{V'_\alpha | \alpha \in \Delta\}$ is a system of neighborhoods of the zero element of \mathcal{L} , where $V'_\alpha = \{L | L \in \mathcal{L}, L(\mathcal{G}) \subseteq V_\alpha\}$ and $\{V_\alpha | \alpha \in \Delta\}$ is the system of convex neighborhoods of the zero element of \mathcal{E} which are such that $V_\alpha = -V_\alpha$. Under this topology \mathcal{L} becomes a locally convex, linear topological space [10, Theorem 17]. This space is also boundedly complete, for if we suppose that $\{L_\gamma | \gamma \in \Gamma\}$ is a bounded fundamental directed set in \mathcal{L} , then for a fixed $g \in \mathcal{G}$, $L_\gamma(g)$ is a fundamental directed set in \mathcal{E} , which is bounded, the same bound serving for all g . But \mathcal{E} is boundedly complete, so there is an element $L_\infty(g)$ in \mathcal{E} such that $L_\gamma(g) \rightarrow L_\infty(g)$ in \mathcal{E} . The function L_∞ thus defined on \mathcal{G} to \mathcal{E} is bounded because $L_\gamma(g)$ is bounded uniformly with respect to γ and g . Finally L_∞ is constructed so that $L_\gamma \rightarrow L_\infty$ in \mathcal{L} .

Let ϕ_a be the transformation of \mathcal{L} into itself, defined by $(\phi_a L)g = L(ag)$. The set $\{\phi_a | a \in \mathcal{G}\}$ will be denoted by Φ . Each ϕ_a is a linear, congruent transformation of \mathcal{L} into itself.

Let I be the function in \mathcal{L} which is such that $I(g) = g$ (the identity). It is certainly bounded because \mathcal{G} is totally bounded [10, Theorem 10].

Put $\mathcal{B} = \text{Cl}(\text{con}(\Phi I))$, so that \mathcal{B} is the closed, convex set in \mathcal{L} generated by the set of all left translates of the function I . We shall prove that \mathcal{B} is compact. To do this it is sufficient to establish that the set ΦI is totally bounded in \mathcal{L} , for then [10, Theorems 10, 11, 12] $\text{Cl}(\text{con}(\Phi I))$ is totally bounded and bounded, and therefore as a closed bounded subset of a boundedly complete space, it is complete. But a totally bounded, complete uniform space is compact [13, p. 111].

Thus we have to show that for any given $\alpha \in \Delta$, we may find a finite set $a_1, \dots, a_n \in \mathcal{G}$ such that to any $a \in \mathcal{G}$, there is an a_j , $1 \leq j \leq n$, such that $I(ag) - I(a_jg) = ag - a_jg \in V_\alpha$ for all $g \in \mathcal{G}$. First we choose V_γ by hypothesis (1) so that $\mathcal{G}V_\gamma \subseteq V_\alpha/3$, because \mathcal{G} is totally bounded. Also we may choose $g_1, \dots, g_m \in \mathcal{G}$ so that $\mathcal{G} \subseteq \bigcup_{i=1}^m [g_i + V_\gamma]$, and then choose V_δ so that $V_\delta g_i \subseteq V_\alpha/3$, $1 \leq i \leq m$, by hypothesis (2). Finally we choose $a_1, \dots, a_n \in \mathcal{G}$ so that $\mathcal{G} \subseteq \bigcup_{j=1}^n [a_j + V_\delta]$. Then for any given $a \in \mathcal{G}$, we may find an a_j , $1 \leq j \leq n$, so that by using a suitable g_i we have that

$$\begin{aligned} ag &= a_jg = ag - ag_i + ag_i - a_jg_i + a_jg_i - a_jg \\ &\in aV_\gamma + V_\delta g_i + a_jV_\gamma \\ &\subseteq [V_\alpha + V_\alpha + V_\alpha]/3 \\ &\subseteq V_\alpha. \end{aligned}$$

Similarly we may define ϕ'_a to be the congruent transformation $(\phi'_a L)g = L(ga)$, $\Phi' = \{\phi'_a \mid a \in G\}$ and $\mathcal{B}' = \text{Cl}(\text{con } (\Phi' I))$. The closed, convex set \mathcal{B}' will be compact if the set $\Phi' I$ is totally bounded, and this is the case if given $\alpha \in \Delta$ we may find $a_1, \dots, a_n \in \mathcal{G}$ so that for any $a \in \mathcal{G}$, $ga - ga_j \in V_\alpha$ for some a_j and all g . But this is easier to establish than the previous case because, choosing V_β so that $\mathcal{G}V_\beta \subseteq V_\alpha$ and then a_1, \dots, a_n so that $\mathcal{G} \subseteq \bigcup_{j=1}^n [a_j + V_\beta]$, then for any given $a \in \mathcal{G}$ we may find an a_j , $1 \leq j \leq n$, so that $ga - ga_j \in \mathcal{G}V_\beta \subseteq V_\alpha$.

We now observe that $\phi_a \mathcal{B} = \mathcal{B}$ and $\phi'_a \mathcal{B}' = \mathcal{B}'$ for all $a \in \mathcal{G}$ because $\phi_a[\phi_b I] = \phi_{ba} I \in \mathcal{B}$ and therefore $\phi_a \text{con } (\Phi I) \subseteq \mathcal{B}$. Since ϕ_a is congruent and therefore continuous, $\phi_a \mathcal{B} \subseteq \mathcal{B} = \phi_a \mathcal{B} \subseteq \phi_a \mathcal{B}$. Similarly with \mathcal{B}' .

We now have all the hypotheses of Theorem 2, which we apply first to \mathcal{B} to obtain a function $P \in \mathcal{B}$ such that $\phi_a P = P$ for all $a \in \mathcal{G}$. That is, $P(ag) = P(g)$, which proves that $P(g)$ is a constant, which we shall denote by p . Also $p \in \text{Cl}(\text{con } \mathcal{G})$. Since P is in \mathcal{B} , then

$$p = P(g) = \lim_{\alpha} \sum_{n=1}^{n_\alpha} \lambda_{\alpha,n} a_{\alpha,n} g, \quad \lambda_{\alpha,n} \geq 0, \quad \sum_{n=1}^{n_\alpha} \lambda_{\alpha,n} = 1,$$

uniformly in g , whence using hypothesis (2)

$$p = P(g) = P(ga) = pa,$$

for all $a \in \mathcal{G}$.

Similarly, applying the theorem to \mathcal{B}' and Φ' , we obtain a p' in $\text{Cl}(\text{con } \mathcal{G})$ so that $p' = ap'$ for all $a \in \mathcal{G}$. Taking the directed limit of convex sums gives us that $p = pp' = p'$, so that $pg = p = gp$ for all $g \in \mathcal{G}$, and $p^2 = p$.

Ergodicity. We apply the results of the previous section to obtain an ergodic theorem. Let \mathcal{E} be the set of linear bounded operators acting on a Banach space \mathcal{X} , in the strong topology of operators. That is, if we put $\alpha = \{x_1, \dots, x_n; \epsilon\}$, $x_1, \dots, x_n \in \mathcal{X}$ and $\epsilon > 0$, and denote the set of all α by Δ , then if we write

$$V_\alpha = \{T \mid T \in \mathcal{E}, |Tx_i| < \epsilon, i = 1, \dots, n\},^2$$

the family $\{V_\alpha \mid \alpha \in \Delta\}$ is a fundamental system of convex neighborhoods of the zero operator in the strong topology.

It is easy to verify that \mathcal{E} is a locally convex, linear topological space. Also \mathcal{E} is boundedly complete, because if $\{T_\gamma \mid \gamma \in \Gamma\}$ is a bounded fundamental directed set in \mathcal{E} , then by the principle of uniform boundedness [15, Theorem 2.12.2] there is an $M > 0$ such

² We use the notation $\| \cdot \|$ for the norm in a Banach space.

that $|T_\gamma| \leq M$ for all $\gamma \in \Gamma$. But $\{T_\gamma x | \gamma \in \Gamma\}$ is a fundamental directed set in \mathcal{X} which is complete, so we may find $T_\infty x$ so that $T_\gamma x \rightarrow T_\infty x$ in \mathcal{X} . It is a routine matter to verify that T_∞ is a linear operator, $|T_\infty| \leq M$, and $T_\gamma \rightarrow T_\infty$ in \mathcal{E} .

We know also that \mathcal{E} is an abstract algebra satisfying the postulates (1) and (2) of Theorem 3, because: (1) if \mathcal{A} is a totally bounded subset of \mathcal{E} , then it is bounded and therefore bounded in norm, that is, $|T| \leq M$ for all T in \mathcal{A} . Thus if $\alpha = \{x_1, \dots, x_n; \epsilon\}$ put $\beta = \{x_1, \dots, x_n; \epsilon/M\}$, from which we have that $\mathcal{A}V_\beta \subseteq V_\alpha$; and (2) if $\alpha = \{x_1, \dots, x_n; \epsilon\}$, putting $\beta = \{Tx_1, \dots, Tx_n; \epsilon\}$ ensures that $V_\beta T \subseteq V_\alpha$.

We shall adopt the following definitions due to Eberlein [4]. A semigroup \mathcal{S} in \mathcal{E} is *ergodic* if there is a directed set of operators $\{T_\gamma | \gamma \in \Gamma\}$ in \mathcal{E} with the properties:

- (a) $T_\gamma x \in \text{Cl}(\text{con } \mathcal{S})x$ for all $\gamma \in \Gamma$ and all $x \in \mathcal{X}$,
- (b) there is an $M > 0$ such that $|T_\gamma| \leq M$ for all $\gamma \in \Gamma$,
- (c) if $T \in \mathcal{S}$, then $\lim_\gamma (TT_\gamma - T) = 0$, $\lim_\gamma (T_\gamma T - T) = 0$, where the limits are in the strong topology.

Also, an element $x \in \mathcal{X}$ is *ergodic* with respect to this ergodic semigroup if there exists a $y = T_\infty x \in \mathcal{X}$ such that $y \in \text{Cl}((\text{con } \mathcal{S})x)$ and $Sy = y$ for all $S \in \mathcal{S}$.

We shall make use of the following theorems which have been developed elsewhere [6].

THEOREM 4. *If \mathcal{T} is a compact topological semigroup [15], and if \mathcal{S} is a subsemigroup of \mathcal{T} which satisfies the pseudo-commutative conditions*

- (i) if $S_1, S_2 \in \mathcal{S}$, then there exist $S_3, S_4 \in \mathcal{S}$ such that $S_1 S_3 = S_2 S_4$,
- (ii) if $S_1, S_2 \in \mathcal{S}$, then there exist $S_3, S_4 \in \mathcal{S}$ such that $S_1 S_2 = S_3 S_1 = S_2 S_4$,

then the set $\mathcal{G} = \bigcap_{S \in \mathcal{S}} \text{Cl}(SS)$ is a nonvacuous topological group.

If we order the semigroup \mathcal{S} by saying that $S_1 > S_2$ whenever there is an S_3 such that $S_1 = S_2 S_3$, then \mathcal{S} is a directed set and \mathcal{G} is its collection of cluster points.

THEOREM 5. *With the same hypotheses as for Theorem 4, if $S \in \mathcal{S}$ and $G \in \mathcal{G}$, then $SG \in \mathcal{G}$ and $GS \in \mathcal{G}$.*

Our main theorem is the following.

THEOREM 6. *Let \mathcal{S} be a totally bounded multiplicative semigroup in \mathcal{E} , which satisfies the pseudo-commutative laws (i) and (ii) of Theorem 4, then \mathcal{S} is an ergodic semigroup and every element of \mathcal{X} is ergodic.*

PROOF. We first show that $\bar{\mathcal{S}}$ is a topological semigroup. Let A and B be elements of $\bar{\mathcal{S}}$. We may choose directed sets $\{A_\alpha | \alpha \in \Delta\}$ and

$\{B_\alpha | \alpha \in \Delta\}$ in \mathfrak{S} , so that $A_\alpha \in A + V_\alpha$ and $B_\alpha \in B + V_\alpha$. Then, $\{A_\alpha B_\alpha | \alpha \in \Delta\}$ is a directed set in \mathfrak{S} which converges to AB because, given V_γ , we may choose V_β so that $\bar{\mathfrak{S}} V_\beta \subseteq V_\gamma/2$, and $V_\beta B \subseteq V_\gamma/2$. Then for $\alpha > \beta$ we have that

$$\begin{aligned} A_\alpha B_\alpha - AB &= A_\alpha B_\alpha - A_\alpha B + A_\alpha B - AB \\ &\in A_\alpha V_\alpha + V_\alpha B \\ &\subseteq (V_\gamma + V_\gamma)/2 \subseteq V_\gamma. \end{aligned}$$

The set $\bar{\mathfrak{S}}$ is therefore a topological semigroup.

The semigroup $\bar{\mathfrak{S}}$ is compact because it is a totally bounded closed set in a boundedly complete topological linear space. By Theorem 4, the set $\mathcal{G} = \bigcap_{S \in \bar{\mathfrak{S}}} \text{Cl}(\mathfrak{S}S)$ is a topological group. By Theorem 3 and because $\mathcal{G} \subseteq \bar{\mathfrak{S}}$, there is an operator $P \in \text{Cl}(\text{con } \mathcal{G})$ such that $PG = P = GP$ for all $G \in \mathcal{G}$. Then by Theorem 5, if $S \in \bar{\mathfrak{S}}$ and $G \in \mathcal{G}$, then SG and GS are in \mathcal{G} and

$$SP = S(GP) = (SG)P = P = P(GS) = (PG)S = PS.$$

We may therefore put $P = T_\gamma$ for all γ and we obtain a trivial directed set, satisfying the conditions required in order that \mathfrak{S} should be an ergodic semigroup.

Finally if $x \in X$, then $y = Px \in \text{Cl}((\text{con } \mathfrak{S})x)$ and $Sy = SPx = Px = y$ for all $S \in \bar{\mathfrak{S}}$, so we have also that every element of X is ergodic.

We may even say a little more about the operator P , as in the following theorem.

THEOREM 7. *Under the hypotheses of Theorem 6, the following statements are equivalent.*

- (1) $P \in \mathcal{G}$.
- (2) $P = E$, the identity of \mathcal{G} .
- (3) \mathcal{G} consists of one element.
- (4) If \mathfrak{S} is directed by saying that $S_1 > S_2$ whenever $S_1 = S_2 S_3$, then \mathfrak{S} converges in \mathcal{E} .
- (5) The directed set \mathfrak{S} converges in \mathcal{E} to $E = P$.

PROOF. (1) implies (2), because $P^2 = P$ and a group can have only one idempotent element which is the identity.

(2) implies (3), because $G = EG = PG = P = E$ for all $G \in \mathcal{G}$.

(3) implies (4), because the directed set \mathfrak{S} in the compact space $\bar{\mathfrak{S}}$ has a unique cluster point E , so that it converges to E .

(4) implies (5), because if the directed set \mathfrak{S} converges, then from the definition of \mathcal{G} its limit is E and \mathcal{G} reduces to one element E . Thus $P \in \text{Cl}(\text{con } \mathcal{G}) = E$.

(5) implies (1) easily.

REFERENCES

1. L. Alaoglu and G. Birkhoff, *General ergodic theorems*, Ann. of Math. vol. 41 (1940) pp. 293–309.
2. M. M. Day, *Ergodic theorems for Abelian semigroups*, Trans. Amer. Math. Soc. vol. 51 (1942) pp. 399–412.
3. ———, *Means for the bounded functions and ergodicity of the bounded representations of semigroups*, Trans. Amer. Math. Soc. vol. 69 (1950) pp. 276–291.
4. W. F. Eberlein, *Abstract ergodic theorems*, Trans. Amer. Math. Soc. vol. 67 (1949) pp. 217–240.
5. S. Kakutani, *Two fixed point theorems concerning bicomact convex sets*, Proc. Imp. Acad. Tokyo vol. 14 (1938) pp. 242–245.
6. J. E. L. Peck, *The embedding of a topological semigroup in a topological group and its generalizations*, unpublished.
7. K. Yosida, *Asymptotic almost periodicities and ergodic theorems*, Proc. Imp. Acad. Tokyo vol. 15 (1939) pp. 255–259.
8. D. H. Hyers, *Linear topological spaces*, Bull. Amer. Math. Soc. vol. 51 (1945) pp. 1–21.
9. A. E. Taylor, *The weak topologies of Banach spaces*, Proc. Nat. Acad. Sci. U.S.A. vol. 25 (1939) pp. 438–440.
10. J. von Neumann, *On complete topological spaces*, Trans. Amer. Math. Soc. vol. 37 (1935) pp. 1–20.
11. G. Birkhoff, *Moore-Smith convergence in general topology*, Ann. of Math. vol. 38 (1937) pp. 39–56.
12. W. Maak, *Eine neue Definition der fastperiodischen Funktionen*, Abh. Math. Sem. Hansischen Univ. vol. 11 (1936) pp. 240–244.
13. N. Bourbaki, *Topologie générale*, chaps. 1 and 2, Paris, 1940.
14. J. W. Tukey, *Convergence and uniformity in topology*, Annals of Mathematics Studies, no. 2, 1940.
15. E. Hille, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloquium Publications, vol. 31, 1948.

YALE UNIVERSITY