ADDENDUM: DERIVATIVES OF INFINITE ORDER

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We gave an incomplete proof that, if \( f(x) \) belongs to a quasi-analytic class \( C[ M_n ] \) in \( a < x < b \) and if \( f^{(n)}(x_0) \to L \) for one \( x_0 \) in \( (a, b) \), then \( f(x) \) is analytic in \( (a, b) \) (and consequently \( f^{(n)}(x) \to Le^{-\alpha n} \) in \( a < x < b \)); the proof was completed by S. Mandelbrojt.\(^2\) We now wish to point out that in fact T. Bang\(^3\) had already shown that if \( f(x) \) belongs to a quasi-analytic class on \( a < x < b \) and \( g(x) \) is analytic, then \( f^{(n)}(x_0) = g^{(n)}(x_0) \) for all \( n \) and \( a < x_0 < b \) implies \( f(x) \equiv g(x) \), which is precisely the result which we needed for our proof.

Bang has also pointed out to us that a function constructed in his thesis\(^4\) answers a question raised by us.\(^5\) We asked whether it is possible to have \( \lim_{n \to \infty} f^{(n)}(x)/\lambda_n = g(x), \) \( a \leq x \leq b, \) with \( \lim \inf \left| \frac{\lambda_{n-1}}{\lambda_n} \right| = 0, \) and \( g(x) \neq 0 \) in \( a < x < b. \) Bang constructed a function \( f(x) \) analytic except for \( x = 0, \) with \( f^{(n)}(0) \) tending to \( \infty \) arbitrarily rapidly; if we take \( \lambda_n = \left| f^{(n)}(0) \right|, \) we have \( g(x) = 0 \) for \( x \neq 0, g(0) = 1. \)

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\(^3\) T. Bang, Om quasi-analytiske Funktioner, Copenhagen thesis, 1946, p. 84.
\(^4\) Bang, loc. cit.
\(^5\) Boas and Chandrasekharan, op. cit., p. 525.