ON THE BEHAVIOR OF FOURIER SINE TRANSFORMS
NEAR THE ORIGIN

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1. On the open half-line \( x > 0 \), let \( f(x) \) be a non-negative, monotone function which tends to 0 as \( x \to \infty \) and behaves, as \( x \to 0 \), in such a way that

\[
\int_{0}^{1} xf(x) \, dx < \infty
\]

(so that \( f(x) \) need not be bounded). It is well known that the improper integral

\[
F(t) = \int_{0}^{\infty} f(x) \sin tx \, dx
\]

must then converge, and that

\[
F(t) > 0, \quad \text{where} \quad 0 < t < \infty,
\]

except when \( f(x) \equiv 0 \). In fact, if \( t > 0 \) is fixed, the integral (2) can be written in the form \( a_0 - a_1 + a_2 - \cdots \), where either \( a_0 = a_1 = \cdots = 0 \) or

\[
a_n \geq a_{n+1} \to 0, \quad \text{hence} \quad \sum_{n=0}^{\infty} (-1)^n a_n > 0.
\]

It seems to be worth observing that, for small \( t \), the assertion of (3) can be refined substantially, since the above assumptions on \( f(x) \) imply that

\[
\lim_{t \to 0} \inf F(t)/t > 0.
\]

What is more, \( F(t)/t \) must tend to a positive limit (\( \leq \infty \)) and the latter can be represented as

\[
\lim_{t \to 0} F(t)/t = \int_{0}^{\infty} xf(x) \, dx,
\]

where it is understood that the integral (6) can have the value \( \infty \).

2. This result has an interesting implication for the Fourier-Stieltjes transforms of certain distribution functions. If, in addition

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to the above properties, $f(x)$ satisfies
\[ \int_{0}^{\infty} f(x) \, dx = 1, \]
let $\delta(x) = 0$ or $f(x)$ according as $x \leq 0$ or $x > 0$; so that
\[ \sigma(x) = \int_{-\infty}^{x} \delta(s) \, ds, \quad \text{where} \quad -\infty < x < \infty, \]
is a distribution function. It follows that the Fourier-Stieltjes transform of $\sigma(x)$,
\[ \Gamma(t) = \int_{0}^{+\infty} e^{i t x} d\sigma(x), \]
has a derivative at $t = 0$ if and only if $\sigma(x)$ has a finite first moment,
\[ \int_{0}^{\infty} x \, d\sigma(x) < \infty. \]
On the other hand, it is known that the existence of this first moment is sufficient, but not necessary, in order that
\[ \int_{0}^{\infty} \cos t x \, d\sigma(x), \]
the real part of $\Gamma(t)$, be differentiable at $t = 0$ (A. Wintner, *The Fourier transforms of probability distributions*, Edwards Brothers, 1947, p. 19; in the example given there, $\delta(x)$ is not monotone for small positive $x$; however, the example is easily altered so as to comply with this condition).

3. Proof of (6). The "alternating" character of the improper integral (2) (cf. (4)) implies that
\[ F(t) \leq \int_{0}^{+t} f(x) \sin tx \, dx. \]
Hence
\[ F(t)/t \leq \int_{0}^{+t} xf(x) \sin tx/tx \, dx \leq \int_{0}^{+t} xf(x) \, dx, \]
since $\sin tx \leq tx$. Consequently, as $t \to +0$,
\[ (7) \quad \lim \sup F(t)/t \leq \int_{0}^{\infty} xf(x) \, dx. \]
It remains to show that, as \( t \to +0 \),

\[
\lim \inf \frac{F(t)}{t} \geq \int_0^\infty x f(x) \, dx.
\]

To this end, it will first be shown that if \( X > 0 \) is arbitrary, then, as \( t \to +0 \),

\[
\lim \inf \frac{F(t)}{t} \geq \int_0^{\infty} x f(x) \, dx - \frac{1}{2} X^2 f(X).
\]

On \( 0 < x < \infty \), define three functions \( f_1(x), f_2(x), f_3(x) \) by placing them respectively equal to \( f(x) \), \( f(X) \), \( f(x) \) or to 0, 0, \( f(x) \) according as \( 0 < x \leq X \) or \( X < x < \infty \). Then \( f_1, f_2, f_3 \) satisfy the same conditions as \( f \), so that their respective sine transforms \( F_1, F_2, F_3 \) are non-negative (for \( t > 0 \)). Since \( f = f_1 - f_2 + f_3 \), it follows that \( F = F_1 - F_2 + F_3 \). Hence, as \( t \to +0 \),

\[
\lim \inf \frac{F(t)}{t} \geq \lim \frac{F_1(t)}{t} - \lim \frac{F_2(t)}{t}.
\]

The existence of the limits on the right-hand side is clear; in fact, since the sine transforms

\[
F_i(t) = \int_0^X f_i(x) \sin x \, dt,
\]

are integrals over a finite interval, it follows that

\[
\frac{F_i(t)}{t} \to \int_0^X x f_i(x) \, dx
\]

as \( t \to +0 \).

The inequality (9) follows from (10), (11) and the definitions of \( f_1, f_2 \).

If \( 0 < \alpha < 1 \), the monotony of \( f(x) \) shows that

\[
\int_\alpha^X x f(x) \, dx \geq f(X) \int_\alpha^X x \, dx = \frac{1}{2} X^2 (1 - \alpha^2) f(X).
\]

Hence the inequality (9) implies that

\[
\lim \inf F(t) / t \geq \int_0^X x f(x) \, dx - \frac{1}{2} \alpha^2 X^2 f(X).
\]

Choose \( \alpha = \alpha(X) \) as a function of \( X \), defined in such a way that \( 0 < \alpha < 1 \), and that \( \alpha X \to \infty \) but \( \alpha^2 X^2 f(X) \to 0 \), as \( X \to \infty \). The existence of such functions \( \alpha = \alpha(X) \) is clear, since \( f(X) \to 0 \) as \( X \to \infty \). Obviously, (8) follows by letting \( X \to \infty \) in (12). This completes the proof of (6).

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