ON THE BEHAVIOR OF FOURIER SINE TRANSFORMS NEAR THE ORIGIN

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1. On the open half-line $x>0$, let $f(x)$ be a non-negative, monotone function which tends to 0 as $x\to \infty$ and behaves, as $x\to 0$, in such a way that

\[ \int_{+0}^{1} xf(x)dx < \infty \]

(so that $f(x)$ need not be bounded). It is well known that the improper integral

\[ F(t) = \int_0^\infty f(x) \sin tx \, dx \]

must then converge, and that

\[ F(t) > 0, \text{ where } 0 < t < \infty, \]

except when $f(x) \equiv 0$. In fact, if $t>0$ is fixed, the integral (2) can be written in the form $a_0 - a_1 + a_2 - \cdots$, where either $a_0 = a_1 = \cdots = 0$ or

\[ a_n \geq a_{n+1} \to 0, \text{ hence } \sum_{n=0}^\infty (-1)^n a_n > 0. \]

It seems to be worth observing that, for small $t$, the assertion of (3) can be refined substantially, since the above assumptions on $f(x)$ imply that

\[ \liminf_{t\to 0} F(t)/t > 0. \]

What is more, $F(t)/t$ must tend to a positive limit ($\leq \infty$) and the latter can be represented as

\[ \lim_{t\to 0} F(t)/t = \int_0^\infty xf(x)dx, \]

where it is understood that the integral (6) can have the value $\infty$.

2. This result has an interesting implication for the Fourier-Stieltjes transforms of certain distribution functions. If, in addition

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to the above properties, \( f(x) \) satisfies
\[ \int_{0}^{\infty} f(x) \, dx = 1, \]

let \( \delta(x) = 0 \) or \( f(x) \) according as \( x \leq 0 \) or \( x > 0 \); so that
\[ \sigma(x) = \int_{-\infty}^{x} \delta(s) \, ds, \quad \text{where} \quad -\infty < x < \infty, \]
is a distribution function. It follows that the Fourier-Stieltjes transform of \( \sigma(x) \),
\[ \Gamma(t) = \int_{0}^{+\infty} e^{itx} \sigma(x), \]
has a derivative at \( t = 0 \) if and only if \( \sigma(x) \) has a finite first moment,
\[ \int_{0}^{\infty} x \sigma(x) < \infty. \]

On the other hand, it is known that the existence of this first moment is sufficient, but not necessary, in order that
\[ \int_{0}^{\infty} \cos tx \sigma(x), \]
the real part of \( \Gamma(t) \), be differentiable at \( t = 0 \) (A. Wintner, *The Fourier transforms of probability distributions*, Edwards Brothers, 1947, p. 19; in the example given there, \( \delta(x) \) is not monotone for small positive \( x \); however, the example is easily altered so as to comply with this condition).

3. **Proof of (6).** The "alternating" character of the improper integral (2) (cf. (4)) implies that
\[ F(t) \leq \int_{0}^{\infty} f(x) \sin tx \, dx. \]
Hence
\[ F(t)/t \leq \int_{0}^{\infty} xf(x)(\sin tx/tx) \, dx \leq \int_{0}^{\infty} xf(x) \, dx, \]
since \( \sin tx \leq tx \). Consequently, as \( t \to +0, \)
\[ \lim \sup F(t)/t \leq \int_{0}^{\infty} xf(x) \, dx. \]
It remains to show that, as $t \to +0$,

$$\lim \inf F(t)/t \geq \int_0^\infty xf(x)dx.$$  

To this end, it will first be shown that if $X > 0$ is arbitrary, then, as $t \to +0$,

$$\lim \inf F(t)/t \geq \int_0^X xf(x)dx - \frac{1}{2} X^2 f(X).$$

On $0 < x < \infty$, define three functions $f_1(x), f_2(x), f_3(x)$ by placing them respectively equal to $f(x), f(X), f(X)$ or to 0, 0, $f(x)$ according as $0 < x \leq X$ or $X < x < \infty$. Then $f_1, f_2, f_3$ satisfy the same conditions as does $f$, so that their respective sine transforms $F_1, F_2, F_3$ are non-negative (for $t > 0$). Since $f = f_1 - f_2 + f_3$, it follows that $F = F_1 - F_2 + F_3$. Hence, as $t \to +0$,

$$\lim \inf F(t)/t \geq \lim F_1(t)/t - \lim F_2(t)/t.$$  

The existence of the limits on the right-hand side is clear; in fact, since the sine transforms $F_i(t) = \int_0^x f_i(x) \sin xt \, dx$, where $i = 1, 2$, are integrals over a finite interval, it follows that

$$F_i(t)/t \to \int_0^X xf_i(x)dx,$$  

as $t \to +0$.

The inequality (9) follows from (10), (11) and the definitions of $f_1, f_2$.

If $0 < \alpha < 1$, the monotony of $f(x)$ shows that

$$\int_0^x xf(x)dx \geq f(X) \int_0^X xdx = \frac{1}{2} X^2(1 - \alpha^2)f(X).$$

Hence the inequality (9) implies that

$$\lim \inf F(t)/t \geq \int_0^\alpha x f(x)dx - \frac{1}{2} \alpha^2 X^2 f(X).$$

Choose $\alpha = \alpha(X)$ as a function of $X$, defined in such a way that $0 < \alpha < 1$, and that $\alpha X \to \infty$ but $\alpha^2 X^2 f(X) \to 0$, as $X \to \infty$. The existence of such functions $\alpha = \alpha(X)$ is clear, since $f(X) \to 0$ as $X \to \infty$. Obviously, (8) follows by letting $X \to \infty$ in (12). This completes the proof of (6).

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