FUNCTIONS WITH PRESCRIBED LIPSCHITZ CONDITION

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1. Introduction. In this note we construct a class of functions each of which satisfies at every point a Lipschitz condition of prescribed order $\alpha$ ($0 < \alpha < 1$). The construction is based on a method given by Knopp [1] for construction of continuous nondifferentiable functions, and follows more particularly a construction of van der Waerden [2].

2. The construction. We first define a fundamental periodic function $g(t, h)$. The function $g(t, h)$ has period $2h$ in $t$, equals zero for even multiples of $h$, equals one for odd multiples of $h$, and is linear between successive multiples of $h$. It is thus a saw-tooth function of $t$.

Now let a number $\alpha$ be given, where $0 < \alpha < 1$. Let $A$ be an integer for which

$$2^{2A(1-\alpha)} > 2.$$ 

The function we are constructing is

$$g(t) = \sum_{n=1}^{\infty} 2^{-2A \alpha n} g(t, 2^{-2A n}).$$

It will be shown that $g(t)$ satisfies for each value of $t$ a Lipschitz condition of order precisely $\alpha$.

3. Proof of Lipschitz condition.

THEOREM. The function $g(t)$ defined above satisfies for each value of $t$ a Lipschitz condition of order precisely $\alpha$. That is, there exist two positive constants $K_1$ and $K_2$ such that

(a) for any $t$ and any $\Delta t$,

$$|\Delta g| < K_1 |\Delta t|^\alpha$$

where $\Delta g = g(t+\Delta t) - g(t)$, and

(b) for any $t$ and for infinitely many, arbitrarily small $\Delta t$,

$$|\Delta g| > K_2 |\Delta t|^\alpha.$$

PROOF. We shall denote the $k$th summand of the series for $g(t)$ by $g_k$. To prove assertion (a) we let $m$ be the integer such that

$$2^{-2A(m+1)} < \Delta t \leq 2^{-2A m}.$$
Since the slope of the linear portions of $g_k(t)$ is $\pm 2^{2A}(1-a)$,

$$|\Delta g_k| \leq 2^{2A} k(1-a) |\Delta t| \leq 2^{2A} (k(1-a) - 2A m) \quad (k \leq m).$$

Since the maximum oscillation of $g_k(t)$ is $2^{-2Aa_k}$,

$$|\Delta g_k| \leq 2^{-2Aa} \quad (k > m).$$

Combining the above,

$$|\Delta g| \leq \sum_{k=1}^{m} 2^{-2A} m 2^{2A} (1-a) k + \sum_{k=m+1}^{\infty} 2^{-2Aa} k$$

$$\leq \frac{2^{2A} (1-a) - 2A m}{2^{2A} (1-a) - 1} + \frac{2^{-2Aa} (m+1)}{1 - 2^{-2Aa}}.$$

Now $|\Delta t| > 2^{-2Aa(m+1)}$. Therefore

$$\frac{|\Delta g|}{|\Delta t|} < \frac{2^{2A}}{2^{2A} (1-a) - 1} + \frac{1}{1 - 2^{-2Aa}},$$

which proves assertion (a).

To prove assertion (b) we use a lemma on geometrical progressions.

**Lemma.** If the ratio in a geometric progression is positive and less than $1/2$, and the first term is 1, then the first term exceeds the sum of the remaining terms by at least

$$\frac{1 - 2r}{1 - r}.$$

**Proof of Lemma.** If the progression is infinite, the sum of the terms following the first is $r/(1-r)$, and if the progression is finite the sum of the terms following the first is less than this. If $r < 1/2$, $r/(1-r) < 1$, so that the first term, 1, exceeds the sum of the remaining terms by at least

$$1 - r/(1 - r) = (1 - 2r)/(1 - r).$$

Turning now to the proof of (b), for a given $t$ let $\Delta t$ be equal numerically to $2^{-2A(m+1)}$

and in a direction so as not to include a multiple of $2^{-2A m}$, where $m$ is any positive integer. For this $\Delta t$,

$$\Delta g_k = 0 \quad (k > m).$$
while

$$\Delta g_k = \pm 2^{2A} k(1-a) 2^{-2A(m+1)}$$

if \( k \leq m \).

Thus

$$\Delta g = 2^{-2A(m+1)} \left[ \pm 2^{2A} m(a-1) \pm 2^{2A} (m-1)(1-a) \pm \ldots \pm 2^{2A} (1-a) \right]$$

$$= 2^{-2A-2A} ma \left[ \pm 1 \pm 2^{-2A} (1-a) \pm 2^{-4A} (1-a) \pm \ldots \right].$$

Now since \( A \) was chosen so that \( 2^{-2A(1-a)} < 1/2 \), we may apply the lemma and conclude that

$$|\Delta g| > 2^{-2A-2A} ma \frac{1 - 2^{-2A} (1-a) \cdot 2}{1 - 2^{-2A} (1-a)}.$$ 

Now \( |\Delta t|^\alpha = 2^{-2A} 2^{-2A} \). Therefore

$$\frac{|\Delta g|}{|\Delta t|^\alpha} > 2^{-2A} (1-a) \frac{1 - 2 \cdot 2^{-2A} (1-a)}{1 - 2^{-2A} (1-a)},$$

and since this is true for any \( m \), assertion (b) is proved.

It should be remarked that for any \( \alpha \) in the range \( 0 < \alpha < 1 \) the function \( g(t) \) is continuous and nondifferentiable.

BIBLIOGRAPHY


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