A COMBINATORIAL THEOREM WITH AN APPLICATION TO LATIN RECTANGLES

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1. **Introduction.** In the present paper a study is made of matrices of r rows and n columns, composed entirely of zeros and ones, with exactly k ones in each row. The problem considered is that of adjoining $n-r$ rows of zeros and ones to obtain a square matrix with exactly k ones in each row and in each column. In §2 it is shown that the obvious necessary conditions for the adjunction of $n-r$ rows are also sufficient. The theorem of §2 has an immediate application to the study of latin squares, and yields in §3 a generalization of the basic existence theorem of Marshall Hall [2].[1]

2. **A combinatorial theorem.**

THEOREM 1. *Let A be a matrix of r rows and n columns, composed entirely of zeros and ones, where $1 \leqq r < n$. Let there be exactly k ones in each row, and let $N(i)$ denote the number of ones in the ith column of A. If, for each $i = 1, 2, \cdots, n$,*

$$k - (n - r) \leqq N(i) \leqq k,$$

then $n-r$ rows of zeros and ones may be adjoined to A to obtain a square matrix with exactly k ones in each row and in each column.

The proof is by mathematical induction. Let t denote the number of columns of A with $N(i) < k$. Then $n-t$ denotes the number of columns of A with $N(i) = k$, and consequently $kr = N(1) + \cdots + N(n) \geqq (n-t)k + (k-(n-r))t$. Thus $k(r-n) \geqq t(r-n)$, whence $t \geqq k$.

Next let p denote the number of columns of A with $N(i) = k - (n-r)$. Then $n-p$ denotes the number of columns with $N(i) > k-(n-r)$. Consequently $kr = N(1) + \cdots + N(n) \leqq p(k-(n-r)) + (n-p)k$, whence $k(r-n) \leqq p(r-n)$ and $p \leqq k$.

We now adjoin to A a row consisting of k ones and $n-k$ zeros. Since $t \geqq k$, there are at least k positions where ones may be inserted so that the resulting $(r+1)$-rowed matrix will have at most k ones in each column. Moreover, since $p \leqq k$, the ones may be inserted in all of those columns with $N(i) = k-(n-r)$. In the resulting $(r+1)$-rowed matrix, let $M(i)$ denote the number of ones in the ith column.

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[1] Numbers in brackets refer to the references at the end of the paper

Because of the structure of the adjoined row, it is clear that

$$k - (n - (r + 1)) \leqq M(i) \leqq k.$$

The process may be continued inductively, and the resulting square matrix possesses k ones in each row and column.

A rectangular matrix L composed of zeros and ones is called a permutation matrix provided that it satisfies the equation $LL^T = I$, where L^T is the transpose of L and I is the identity matrix. Let A be a square matrix of zeros and ones, with exactly k ones in each row and in each column. A classical theorem of König asserts that

$$A = L_1 + L_2 + \cdots + L_k,$$

where the L_i are permutation matrices [**5**]. Actually König's theorem is a special case of P. Hall's theorem on the distinct representatives of subsets [**4**]. The latter theorem has been the subject of the recent investigations of Everett and Whaples [**1**], and Marshall Hall [**3**].

COROLLARY. *For the matrix A of Theorem* 1, $A = L_1 + L_2 + \cdots + L_k$, *where the* L_i *are permutation matrices.*

The corollary follows immediately upon and application of Theorem 1 and König's theorem.

3. **The application to latin rectangles.** A latin rectangle of order r by s based upon the integers $1, 2, \cdots, n$ is defined as an array of r rows and s columns formed from the integers $1, 2, \cdots, n$ in such a way that the integers in each row and in each column are distinct. The latin rectangle is said to be extendible to an n by n latin square provided that it is possible to adjoin $n-s$ columns and $n-r$ rows in such a way that the resulting array is an n by n latin square. By utilizing the theory of distinct representatives of subsets, Marshall Hall has shown that every r by n latin rectangle may be extended to an n by n latin square [**2**].

THEOREM 2. *Let T be an r by s latin rectangle based upon the integers* $1, 2, \cdots, n$. *Let* $N(i)$ *denote the number of times that the integer i occurs in T. A necessary and sufficient condition in order that T may be extended to an n by n latin square is that for each* $i = 1, 2, \cdots, n$,

$$N(i) \geqq r + s - n.$$

Let T_i denote the set of s integers formed from the ith row of T. Let S_i denote the set of the $k = n - s$ integers among $1, 2, \cdots, n$ which are not in T_i, and let $M(i)$ denote the number of times that the integer i occurs among the sets $S_1, S_2, \cdots, S_r$.

Now if T is extendible to a latin square, then the integer i cannot occur among the sets $S_1, S_2, \cdots, S_r$ more than $k=n-s$ times. Hence $M(i) \leqq n-s$. But $N(i)+M(i)=r$, whence $N(i) \geqq r+s-n$. Thus the condition of the theorem is a necessary one.

To prove the sufficiency we form from the sets S_i a matrix A of order r by n, composed of zeros and ones. Let S_i be composed of the integers $i_1, i_2, \cdots, i_k$. In the ith row of A insert ones in columns $i_1, i_2, \cdots, i_k$, and zeros elsewhere in this row. The matrix A has then exactly k ones in each row, and $M(i)$ is now the sum of the ith column of A. By hypothesis $N(i)=r-M(i) \geqq r+s-n$, so that for $i=1, 2, \cdots, n$, $M(i) \leqq k$. Since T is an r by s latin rectangle, $N(i) \leqq s$, whence $k-(n-r) \leqq M(i)$. By the corollary of Theorem 1, it now follows that $A=L_1+L_2+\cdots+L_k$, where the L_t are rectangular permutation matrices. Let the one in row j of L_t occur in column t_j. From the integers t_j form the k sets $(t_1, t_2, \cdots, t_r)$, each containing r distinct integers. These sets may now be adjoined to T to obtain a latin rectangle of order r by n. The latter may then be extended to an n by n latin square as in [2]. This does not differ essentially from completing the transposed n by r latin rectangle to an n by n latin square by the method already indicated, the condition on $N(i)$ being then trivially satisfied.

References

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