NIL PI-RINGS

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Let \( S \) be a ring which satisfies a polynomial identity (in short: a PI-ring). The ring \( S \) will be said to be a PI-ring of degree \( d \) if \( d \) is the minimal degree of the polynomial identities satisfied by \( S \). We denote by \( N = N(S) \) the radical of \( S \), that is, the sum of all nilpotent ideals of \( S \). Levitzki [1] has proved that a nil PI-ring of degree \( d \) is an L-ring (that is, it coincides with its lower radical) and its length is bounded by \( \log d / \log 2 \). In the present note we show that the length of a nil PI-ring is not greater than 2 and that nil PI-rings of length 2 really exist. Even more, if \( S \) is a nil PI-ring of degree \( d \) then \( S/N \) is a nilpotent ring whose index is bounded by \( [d/2] \). This is a direct consequence of the following generalization of [1, Theorem 1]:

**Theorem 1.** If \( S \) is a PI-ring of degree \( d \) and \( T \) is a nil subring of \( S \), then \( T^m \subseteq N \) where \( m = [d/2] \).

**Proof.** First we consider the case where \( T \) is a nilpotent subring of \( S \). The proof of this case differs from the proof of [1, Theorem 1] only in that we consider a nilpotent subring instead of a single nilpotent element. That is, we consider the following subrings of \( S \):

\[ A_{2i-1} = T^{n-i}ST^{i-1}, \]
\[ A_{2i} = T^{n-i}ST^i, \]
where \( n \) is an integer greater than \( m = [d/2] \).

It is readily seen that \( A_\lambda A_\mu \subseteq ST^nS \) if \( \lambda > \mu \), hence

\[ A_{i_1}A_{i_2} \cdots A_{i_d} \subseteq ST^nS, \]
if \((i_1, \cdots, i_d)\) is a permutation of the \( d \) letters \( 1, 2, \cdots, d \) which is not the identical permutation.

By (1) it follows also that

\[ A_1A_2 \cdots A_d = (T^{n-1}S)^dT^m. \]

We may assume by [1, Lemma 3] that \( S \) satisfies the following identity:4

\[^4\text{Received by the editors June 19, 1950.}
\[^1\text{Numbers in brackets refer to the bibliography at the end of the paper.}
\[^2\text{It is not assumed that \( S \) is a nil ring.}
\[^3\text{I am indebted to Levitzki for the present proof of this case.}
\[^4\text{Compare with (10) of [1]. For the conditions satisfied by the coefficients of the identity see [1, p. 335].} \]
(4) \[ x_1x_2 \cdots x_d = \sum \beta^{-1}(i) x_i \cdots x_{i_d}, \]
where the sum ranges over all permutations (i) of d letters, except the identical permutation.

From (2), (3), and from condition (II) of [1] satisfied by the coefficients of the identity (1), it follows by substituting \( x_i = a_i \) in (4) where \( a_i \) ranges over all elements of \( A_i, i = 1, \ldots, d, \) that

\[ (T^{n-1}S)^d T^m \subseteq ST^S. \]

Since \( T \) is nilpotent, there exists a smallest exponent \( n \) such that \( ST^S \) is a nilpotent ideal. Suppose \( n > m \), then by (5) it follows easily that \( (ST^{n-1}S)^{d+1} \subseteq ST^S \), hence \( ST^{n-1}S \) is also nilpotent which is a contradiction to the minimality of \( n \). This completes the proof of the theorem in the case of nilpotent subrings.

We turn now to the general case. Let \( T \) be a nil subring of \( S \). By [1, Theorem 1] it follows that the quotient ring \( (T, N)/N \) satisfies an identity of the form \( x^m = 0 \), hence \( (T, N)/N \) is semi-nilpotent (Kaplansky [3, Theorem 5] and Levitzki [1]). Since \( N \) is semi-nilpotent, the subring \( (T, N) \) of \( S \) is also semi-nilpotent. Let \( t_1, \ldots, t_m \) be any \( m \) elements of \( T \), then the semi-nilpotency of \( T \) implies that the ring \( \{t_1, \ldots, t_m\} \) generated by these elements is nilpotent, hence by the preceding case \( \{t_1, \ldots, t_m\}^m \subseteq N \). Thus \( t_1 \cdot t_2 \cdots t_m \subseteq N \). Since this holds for any arbitrarily chosen elements of \( T \), we have \( T^m \subseteq N \). q.e.d.

Remark. By the preceding proof it follows that if \( T \) is a nilpotent subring of \( S \) of index \( p > m \), then \( ST^S \) is a nilpotent ideal in \( S \). A more detailed application of (5) shows that \( 1 + d + \cdots + d^p - m \) is an upper bound for the index of \( ST^S \). Indeed by (5) we have \( (ST^S)^{d+1} \subseteq ST^{m+1}S \) and for the same reason \( (ST^{m+1}S)^{d+1} \subseteq ST^{m+2}S \); hence \( (ST^S)^{1 + d + d^2} \subseteq ST^{n+3}S \). By a successive application of (5), we obtain

\[ (ST^S)^{1 + d + \cdots + d^p - m} \subseteq ST^S = 0, \]

which proves the remark.

By the preceding theorem, we have the following corollary.

Corollary. If \( S \) is a PI-ring of degree \( d \) such that its radical \( N \) is a nilpotent ideal of index \( p \), then the nil subrings of \( S \) are nilpotent rings of index not greater than \( p \lfloor d/2 \rfloor \).

Remark. It has been shown recently [2] that the total matrix algebra of order \( n^2 \) over a commutative field is a PI-ring of degree
Hence by the preceding corollary it follows that the nil subrings of such algebras are nilpotent rings of index less than or equal to \( n \). This is a special case of the well known result concerning nil subrings of rings which satisfy both chain conditions.

A simple consequence of Theorem 1 is:

**Theorem 2.** If \( S \) is a nil PI-ring of degree \( d \), then \( S/N \) is a nilpotent ring whose index is bounded by \( [d/2] \).

This implies that in this case \( S \) is a nil ring, \( S = N_2(S) \), that is:

**Corollary.** A nil PI-ring is an L-ring of length less than or equal to 2.

We conclude with an example of a nil PI-ring \( S \) of degree \( 2n \) such that \( S/N \) is a nilpotent ring whose index is \( n \). This example shows that Theorem 2 provides a complete solution of the problem of the length of nil PI-rings and their structure modulo their radical.

We construct our example as follows: Let \( R \) be a commutative ring with a unit such that its radical \( N(R) \) is not nilpotent. Denote by \( c_{ik}, i, k = 1, 2, \ldots, n \), an orthogonal base of a total matric algebra \( R_n \) of order \( n^2 \) over \( R \). One can easily generalise [2, Theorem 1] to total matric algebras over commutative rings and thus one obtains the result that \( R_n \) satisfies a polynomial identity of degree \( 2n \) (that is, the standard identity \( S_{2n}(x) = 0 \)). Our required ring \( S \) is defined as the totality of the matrices \( \sum \alpha_{ik}c_{ik} \), where \( \alpha_{ik} \in N(R) \) for \( i \geq k \). No restriction is imposed on the elements \( \alpha_{ik}, i < k \), except that \( \alpha_{ik} \in R \).

Since \( S \subseteq R_n \), it follows that \( S \) is a PI-ring of degree less than or equal to \( 2n \). It is readily verified that \( S^n \subseteq N(R)_n \subseteq N(S) \) where \( N(R)_n \) is the totality of the matrices \( \sum \beta_{ik}c_{ik} \), \( \beta_{ik} \in N(R) \). Now consider the element \( c = c_{12} + c_{23} + \cdots + c_{n-1,n} \). The ideals \( c^iS, i = 1, 2, \ldots, n-1 \), contain a ring isomorphic to \( N(R) \), that is, the ring of the matrices \( c\tau = \rho c_{i1} \) where \( \tau = c_{i+1,i+1}, \rho \in N(R) \). The latter ring is not nilpotent, hence \( c^i \in N(S) \). This implies by [1, Theorem 1] that the degree of \( S \) is greater than or equal to \( 2n \). This completes the proof that the ring \( S \) has the required properties.

**Bibliography**


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