AN EXTENSION OF THE "PRINCIPAL THEOREM" OF WEDDERBURN

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1. Introduction. The so-called "principal theorem" of Wedderburn may be stated for an associative algebra, or more generally [3, p. 606]¹ as follows for an alternative algebra over an arbitrary field.

**Theorem 1.** If $\mathfrak{A}$ is an alternative algebra with radical $\mathfrak{N}$ such that the difference algebra $\mathfrak{A}/\mathfrak{N}$ is separable, then there is an algebra $\mathfrak{B}$ such that

\[ \mathfrak{A} = \mathfrak{B} + \mathfrak{N}, \quad \mathfrak{B} \cong \mathfrak{A}/\mathfrak{N}. \]

By considering elements whose principal traces are zero instead of nilpotent elements, this paper exhibits an ideal $\mathfrak{I}$ called the *liberal*, which is a generalization of the radical and which, in fact, reduces to the radical when and only when $\mathfrak{A}/\mathfrak{N}$ is separable. An extension of Theorem 1 is obtained in which there is always a decomposition of type (1) for the liberal of an arbitrary alternative algebra.

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2. The first liberal. Let $\xi = \sum \xi_i e_i$ be a general element [2, p. 112] of an alternative algebra with basis $e_1, e_2, \ldots, e_n$ over an arbitrary field $\mathbb{F}$. Then $T(\xi)$, the negative of the coefficient of the second highest power of $\lambda$ in the minimum polynomial $m(\xi, \lambda)$ of $\xi$, is called the *principal trace* of $\xi$. When the indeterminates $\xi_i$ are replaced by elements $x_i$ of $\mathbb{F}$ in $T(\xi)$ we get the principal trace $T(x)$ of $x = \sum x_i e_i$. The quantity $T(x)$ is independent of the basis of $\mathfrak{A}$.

We now wish to prove the following lemma.

**Lemma 1.** For all $x, y, z$ in an alternative algebra $\mathfrak{A}$ and all $\alpha$ in the base field $\mathbb{F}$ of $\mathfrak{A}$, $T(x+y) = T(x) + T(y)$, $T(\alpha x) = \alpha T(x)$, $T(xy) = T(yx)$ and $T(x \cdot yz) = T(xy \cdot z)$.

Since $\xi$ is also a general element of $\mathfrak{A}_k$ for any extension $k$ of $\mathbb{F}$, we may assume that $\mathbb{F}$ is algebraically closed. Then if $\mathfrak{A}$ is simple it is either associative or the Cayley-Dickson algebra with divisors of zero over $\mathbb{F}$. Lemma 1 is well known for the associative case and is

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¹ Numbers in brackets refer to the references at the end of the paper.

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easily verified for the Cayley-Dickson algebra with divisors of zero over any field.

Now suppose \( A \) is not simple and adjoin a unit element to \( A \) if \( A \) does not already have one. We denote this new algebra by \( A^* \) or if \( A \) already has a unit element we let \( A = A^* \). We shall consider only those elements of \( A^* \) which are in \( A \). Let \( a \xi = aR_1 \) where \( R_1 \) is a right multiplication of \( \xi \) in \( A \). Then the correspondence \( x \rightarrow R_x \) is (1-1) and \( m(\xi, \lambda) \) is the minimum function of \( R_1 \).

Let \( \mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \cdots \oplus \mathcal{S}_k \) be an expression for \( A/\mathfrak{A} \) as a direct sum of simple algebras. Then if the bars denote residue classes modulo \( \mathfrak{A} \), a basis for \( A \) can be so chosen that \( \xi = \xi^{(1)} + \xi^{(2)} + \cdots + \xi^{(k)} \) where \( \xi^{(i)} \) is a general element of \( \mathcal{S}_i \), and \( R_1 \) has the form

\[
(2) \quad R_1 = \begin{bmatrix} X & P \\ 0 & Q \end{bmatrix},
\]

where either \( X = X^{(1)} \oplus X^{(2)} \oplus \cdots \oplus X^{(k)} \) for right multiplications \( X^{(i)} \) of the general elements \( \xi^{(i)} \) of \( \mathcal{S}_i \) or (in case \( A \neq A^* \)) \( X \) is this matrix bordered by \( (\xi_1, \xi_2, \cdots, \xi_n) \) above and a column of zeros on the left.

In either case \( X^{(1)} \oplus X^{(2)} \oplus \cdots \oplus X^{(k)} \) will certainly satisfy the equation \( m'(\lambda) = \prod m_i(\lambda) = 0 \) where \( m_i(\lambda) \) is a minimum polynomial of the algebra \( \mathcal{S}_i \). Hence \( R_1 m'(R_1) \) will be of the form

\[
(3) \quad R_u = \begin{bmatrix} 0 & P' \\ 0 & Q' \end{bmatrix},
\]

for \( u \in \mathfrak{A} \). Therefore \( R_1 \) will satisfy the equation \( [\lambda m'(\lambda)]^r = 0 \) for some integer \( r \). Consequently \( m(\xi, \lambda) \) must divide \( [\lambda m'(\lambda)]^r \) and since the \( m_i(\lambda) \) are irreducible, \( T(\xi) \) must be a sum containing solely certain of the principal traces \( T(\xi^{(i)}) \) of the simple algebras \( \mathcal{S}_i \).

Thus the proof is reduced to the case of a simple algebra and is therefore complete.

It follows from Lemma 1 that the set of all \( x \) of \( A \) such that \( T(xy) = 0 \) for every \( y \) of \( A \) is an ideal \( \mathfrak{T}_1 \) which we shall call the first liberal of \( A \). Since \( T(xy) = 0 \) for all \( x \in \mathfrak{A} \) and all \( y \) in \( A \), it follows that \( \mathfrak{T}_1 \supseteq \mathfrak{A} \) and hence \( \mathfrak{A} \) is also the radical of \( \mathfrak{T}_1 \).

**Theorem 2.** If \( A \) is an alternative algebra with first liberal \( \mathfrak{T}_1 \), then \( T(x) = 0 \) for all \( x \) in \( \mathfrak{T}_1 \).

If \( \mathfrak{T}_1 \) contains some element \( u_1 \) such that \( T(u_1) \neq 0 \), then, because of the linearity of the principal trace function (Lemma 1), a basis \( u_1, u_2, \cdots, u_p \) can be so chosen for \( \mathfrak{T}_1 \) that \( T(u_i) = 0, i > 1 \). Then the set \( \mathfrak{T}_1^* = (u_1, u_2, \cdots, u_p) \) is an ideal of \( \mathfrak{T}_1 \) and \( \mathfrak{T}_1/\mathfrak{T}_1^* \) is a zero alge-
bra of order one. But $\mathfrak{L}/\mathfrak{L}_1$ is semi-simple, a contradiction, and the theorem is proved.

For a given basis $e_1, e_2, \ldots, e_n$ of $\mathfrak{A}$, the matrix $D = T(e_r e_s)$ where $r$ is the row and $s$ is the column in which the element $e_r e_s$ stands is a **discriminant matrix** of $\mathfrak{A}$ and its determinant is a **discriminant** of $\mathfrak{A}$. If the basis is changed to some other basis by a nonsingular transformation of matrix $M$ then the $D$ is changed to $MD_1 M^T$ where $M^T$ is the transpose of $M$. Hence the rank of $D_1$ is invariant under change of basis. Moreover since $e_1, e_2, \ldots, e_n$ is also a basis for $\mathfrak{A}_2$, $D_1$ is also a discriminant matrix of $\mathfrak{A}_2$.

The following lemma can be found in [1, p. 34] for an associative algebra. The proof given here is much the same as for the associative case.

**Lemma 2.** The discriminants of an alternative algebra $\mathfrak{A}$ are different from zero if and only if $\mathfrak{A}$ is separable.

If $\mathfrak{A}$ is not separable there is an extension $\mathfrak{B}$ of $\mathfrak{A}$ such that $\mathfrak{A}_2$ has a nonzero radical. Then $D_1$ can be taken to have the form

$$
\begin{bmatrix}
D_1' & 0 \\
0 & 0
\end{bmatrix}
$$

Hence $|D_1| = 0$. Conversely if $\mathfrak{A}$ is separable it is a direct sum of simple algebras and the proof is reduced to the case where $\mathfrak{A}$ is simple. But then there is an extension $\mathfrak{B}$ of $\mathfrak{A}$ such that $\mathfrak{A}_2$ is either associative or the unique Cayley-Dickson algebra with divisors of zero over $\mathfrak{B}$. The associative case is proved in the reference and it can easily be verified that $|D_1|$ can be taken to be one for the Cayley-Dickson algebra with divisors of zero. Hence $|D_1| \neq 0$.

**Theorem 3.** The order of the first liberal of an alternative algebra $\mathfrak{A}$ is equal to the nullity of the discriminant matrices of $\mathfrak{A}$.

Since $D_1$ is a symmetric matrix by Lemma 1, a nonsingular $M$ can be chosen so that $MD_1 M^T$ is a matrix of type (4) with a nonsingular matrix in the upper left-hand corner. It is then evident that the order of $\mathfrak{L}_1$ is equal to the nullity of $D_1$.

The following corollary then follows immediately from Lemma 2.

**Corollary 3.1.** The first liberal of $\mathfrak{A}$ is zero if and only if $\mathfrak{A}$ is separable.

Hence we have the corollary.
Corollary 3.2. If $A$ is simple, then $T_1 = A$ if and only if $A$ is not separable.

And since a semi-simple algebra is a direct sum of simple algebras we obtain the corollary.

Corollary 3.3. If $A$ is semi-simple, the first liberal is the direct sum of all inseparable components of $A$.

For the basis $e_1, e_2, \ldots, e_n$ of $A$, let $e_{n+1}, \ldots, e_m$ be a basis for $T_1$. It then follows from Theorem 3 and the fact that $e_1, e_2, \ldots, e_n$ is also a basis for $A$ that $(T_1)_R$ is the first liberal of $A$.

Theorem 4. If $T_1$ is the first liberal of an alternative algebra $A$, then $A/T_1$ is separable.

Since $T_1 \supseteq R$, $A/T_1$ is semi-simple. If $R$ is any extension of $R$, then $(A/T_1)_R = A/R/(T_1)_R$ is semi-simple for the same reason and hence $A/T_1$ is separable.

3. The liberal. Before defining the liberal we first prove a useful lemma.

Lemma 3. If $B$ is an ideal of an alternative algebra $A$ such that $A/B$ is separable, then there is an algebra $G$ such that

$$A = B + G, \quad G \cong A/B.$$ 

If $R$ is the radical of $A$, then $B \supseteq R$. Let $A_0 = A/R$, $B_0 = B/R$ so that $B_0$ is an ideal of $A_0$. Since $A_0$ is semi-simple so is $B_0$, and $A_0 = B_0 \oplus C_0$ where $C_0 \cong A_0/B_0$. But $A_0/B_0 \cong A/B$ so $C_0$ is separable. Let $C$ be the sub-algebra of $A$ such that $C/R = C_0$. Then $R$ is the radical of $C$ and by Theorem 1 there is an algebra $G$ such that $C = G + R$, $G \cong C_0$. Now the intersection $D$ of $B$ and $G$ is an ideal of $B$ and hence is semi-simple. But the intersection of $B_0$ and $C_0$ is zero, so $B \subseteq R$. Hence $D = 0$ and the order of $B$ plus the order of $G$ is the order of $A$ so that $A$ is the supplementary sum $B + G$ where $G \cong A/B$.

Let $R$ be the radical of an arbitrary alternative algebra $A$ and let $T'$ be the first liberal of the semi-simple algebra $A/R$. Then by Corollary 3.3, $T'$ is the direct sum of all inseparable components of $A/R$, and we have $A/R = T' \oplus G'$ where $G'$ is the direct sum of all separable components of $A/R$. The unique ideal $T \supseteq T_1$ of $A$ such that $T/R = T'$ is then the minimum ideal of $A$ such that $A/T$ is separable. We shall call $T$ the liberal of $A$. From this definition we see that the liberal reduces to the radical if and only if $A/R$ is separable. Also from Theorem 4 we conclude that $T \subseteq T_1$ and hence $T(x) = 0$ for all $x$ in $T$.

By Lemma 3 we have the following theorem.
Theorem 5. If $\mathcal{X}$ is the liberal of an arbitrary alternative algebra $\mathfrak{A}$, then there is an algebra $\mathfrak{S}$ such that

$$\mathfrak{A} = \mathfrak{S} + \mathcal{X}, \quad \mathfrak{S} \cong \mathfrak{A}/\mathcal{X}. \quad (5)$$

Since $\mathcal{X} = \mathfrak{N}$ when $\mathfrak{A}/\mathfrak{N}$ is separable, the decomposition $(5)$ is the same as $(1)$ when Theorem 1 holds. Moreover there is always a decomposition $(5)$ even when there is no decomposition $(1)$. This decomposition $(5)$ may be trivial of course, that is, when $\mathcal{X} = \mathfrak{A}$, but then there is no proper ideal $\mathfrak{I}$ of $\mathfrak{A}$ such that $\mathfrak{A}/\mathfrak{I}$ is separable. Examples can easily be constructed in which there are nontrivial decompositions $(5)$ and no decomposition $(1)$. It is also true that the liberal and the first liberal do not always coincide.

References


The University of Wisconsin and
The University of Pennsylvania