AN EXTENSION OF THE "PRINCIPAL THEOREM" OF WEDDERBURN

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1. **Introduction.** The so called "principal theorem" of Wedderburn may be stated for an associative algebra, or more generally [**3**, p. 606][1] as follows for an alternative algebra over an arbitrary field.

THEOREM 1. *If $\mathfrak{A}$ is an alternative algebra with radical $\mathfrak{N}$ such that the difference algebra $\mathfrak{A}/\mathfrak{N}$ is separable, then there is an algebra $\mathfrak{B}$ such that*

$$\mathfrak{A} = \mathfrak{B} + \mathfrak{N}, \qquad \mathfrak{B} \cong \mathfrak{A}/\mathfrak{N}. \tag{1}$$

By considering elements whose principal traces are zero instead of nilpotent elements, this paper exhibits an ideal $\mathfrak{T} \supseteq \mathfrak{N}$ called the *liberal*, which is a generalization of the radical and which, in fact, reduces to the radical when and only when $\mathfrak{A}/\mathfrak{N}$ is separable. An extension of Theorem 1 is obtained in which there is always a decomposition of type (1) for the liberal of an arbitrary alternative algebra.

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2. **The first liberal.** Let $\xi = \sum \xi_i e_i$ be a *general element* [**2**, p. 112] of an alternative algebra with basis $e_1, e_2, \cdots, e_n$ over an arbitrary field $\mathfrak{F}$. Then $T(\xi)$, the negative of the coefficient of the second highest power of λ in the *minimum polynomial* $m(\xi, \lambda)$ of ξ, is called the *principal trace* of ξ. When the indeterminates ξ_i are replaced by elements x_i of $\mathfrak{F}$ in $T(\xi)$ we get the *principal trace* $T(x)$ of $x = \sum x_i e_i$. The quantity $T(x)$ is independent of the basis of $\mathfrak{A}$.

We now wish to prove the following lemma.

LEMMA 1. *For all x, y, z in an alternative algebra $\mathfrak{A}$ and all α in the base field $\mathfrak{F}$ of $\mathfrak{A}$, $T(x+y) = T(x) + T(y)$, $T(\alpha x) = \alpha T(x)$, $T(xy) = T(yx)$ and $T(x \cdot yz) = T(xy \cdot z)$.*

Since ξ is also a general element of $\mathfrak{A}_{\mathfrak{K}}$ for any extension $\mathfrak{K}$ of $\mathfrak{F}$, we may assume that $\mathfrak{F}$ is algebraically closed. Then if $\mathfrak{A}$ is simple it is either associative or the Cayley-Dickson algebra with divisors of zero over $\mathfrak{F}$. Lemma 1 is well known for the associative case and is

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[1] Numbers in brackets refer to the references at the end of the paper.

easily verified for the Cayley-Dickson algebra with divisors of zero over any field.

Now suppose $\mathfrak{A}$ is not simple and adjoin a unit element to $\mathfrak{A}$ if $\mathfrak{A}$ does not already have one. We denote this new algebra by $\mathfrak{A}^*$ or if $\mathfrak{A}$ already has a unit element we let $\mathfrak{A}=\mathfrak{A}^*$. We shall consider only those elements of $\mathfrak{A}^*$ which are in $\mathfrak{A}$. Let $a\xi=aR_\xi$ where R_ξ is a right multiplication of ξ in $\mathfrak{A}$. Then the correspondence $x\rightarrow R_x$ is (1-1) and $m(\xi, \lambda)$ is the minimum function of R_ξ.

Let $\mathfrak{S}_1\oplus\mathfrak{S}_2\oplus \cdots \oplus\mathfrak{S}_k$ be an expression for $\mathfrak{A}/\mathfrak{N}$ as a direct sum of simple algebras. Then if the bars denote residue classes modulo $\mathfrak{N}$, a basis for $\mathfrak{A}$ can be so chosen that $\bar{\xi}=\bar{\xi}^{(1)}+\bar{\xi}^{(2)}+ \cdots +\bar{\xi}^{(k)}$ where $\bar{\xi}^{(i)}$ is a general element of $\mathfrak{S}_i$, and R_ξ has the form

$$R_\xi = \begin{bmatrix} X & P \\ 0 & Q \end{bmatrix}, \tag{2}$$

where either $X=X^{(1)}\oplus X^{(2)}\oplus \cdots \oplus X^{(k)}$ for right multiplications $X^{(i)}$ of the general elements $\bar{\xi}^{(i)}$ of $\mathfrak{S}_i$ or (in case $\mathfrak{A}\neq\mathfrak{A}^*$) X is this matrix bordered by $(\xi_1, \xi_2, \cdots, \xi_n)$ above and a column of zeros on the left.

In either case $X^{(1)}\oplus X^{(2)}\oplus \cdots \oplus X^{(k)}$ will certainly satisfy the equation $m'(\lambda)=\prod m_i(\lambda)=0$ where $m_i(\lambda)$ is a minimum polynomial of the algebra $\mathfrak{S}_i$. Hence $R_\xi m'(R_\xi)$ will be of the form

$$R_u = \begin{bmatrix} 0 & P' \\ 0 & Q' \end{bmatrix}, \tag{3}$$

for u in $\mathfrak{N}$. Therefore R_ξ will satisfy the equation $[\lambda m'(\lambda)]^r=0$ for some integer r. Consequently $m(\xi, \lambda)$ must divide $[\lambda m'(\lambda)]^r$ and since the $m_i(\lambda)$ are irreducible, $T(\xi)$ must be a sum containing solely certain of the principal traces $T(\bar{\xi}^{(i)})$ of the simple algebras $\mathfrak{S}_i$.

Thus the proof is reduced to the case of a simple algebra and is therefore complete.

It follows from Lemma 1 that the set of all x of $\mathfrak{A}$ such that $T(xy)=0$ for every y of $\mathfrak{A}$ is an ideal $\mathfrak{T}_1$ which we shall call the *first liberal* of $\mathfrak{A}$. Since $T(xy)=0$ for all x in $\mathfrak{N}$ and all y in $\mathfrak{A}$, it follows that $\mathfrak{T}_1\supseteq\mathfrak{N}$ and hence $\mathfrak{N}$ is also the radical of $\mathfrak{T}_1$.

THEOREM 2. *If $\mathfrak{A}$ is an alternative algebra with first liberal $\mathfrak{T}_1$, then $T(x)=0$ for all x in $\mathfrak{T}_1$.*

If $\mathfrak{T}_1$ contains some element u_1 such that $T(u_1)\neq 0$, then, because of the linearity of the principal trace function (Lemma 1), a basis $u_1, u_2, \cdots, u_p$ can be so chosen for $\mathfrak{T}_1$ that $T(u_i)=0$, $i>1$. Then the set $\mathfrak{T}_1^*=(u_2, u_3, \cdots, u_p)$ is an ideal of $\mathfrak{T}_1$ and $\mathfrak{T}_1/\mathfrak{T}_1^*$ is a zero algebra of order one. But $\mathfrak{T}_1^* \supseteq \mathfrak{N}$ so $\mathfrak{T}_1/\mathfrak{T}_1^*$ is semi-simple, a contradiction, and the theorem is proved.

For a given basis $e_1, e_2, \cdots, e_n$ of $\mathfrak{A}$, the matrix $D_1 = T(e_r e_s)$ where r is the row and s is the column in which the element $e_r e_s$ stands is a *discriminant matrix* of $\mathfrak{A}$ and its determinant is a *discriminant* of $\mathfrak{A}$. If the basis is changed to some other basis by a nonsingular transformation of matrix M then the D_1 is changed to $MD_1\overline{M}$ where $\overline{M}$ is the transpose of M. Hence the rank of D_1 is invarient under change of basis. Moreover since $e_1, e_2, \cdots, e_n$ is also a basis for $\mathfrak{A}_{\mathfrak{K}}$, D_1 is also a discriminant matrix of $\mathfrak{A}_{\mathfrak{K}}$.

The following lemma can be found in [1, p. 34] for an associative algebra. The proof given here is much the same as for the associative case.

LEMMA 2. *The discriminants of an alternative algebra $\mathfrak{A}$ are different from zero if and only if $\mathfrak{A}$ is separable.*

If $\mathfrak{A}$ is not separable there is an extension $\mathfrak{K}$ of $\mathfrak{F}$ such that $\mathfrak{A}_{\mathfrak{K}}$ has a nonzero radical. Then D_1 can be taken to have the form

$$\begin{bmatrix} D_1' & 0 \\ 0 & 0 \end{bmatrix}. \tag{4}$$

Hence $|D_1| = 0$. Conversely if $\mathfrak{A}$ is separable it is a direct sum of simple algebras and the proof is reduced to the case where $\mathfrak{A}$ is simple. But then there is an extension $\mathfrak{K}$ of $\mathfrak{F}$ such that $\mathfrak{A}_{\mathfrak{K}}$ is either associative or the unique Cayley-Dickson algebra with divisors of zero over $\mathfrak{K}$. The associative case is proved in the reference and it can easily be verified that $|D_1|$ can be taken to be one for the Cayley-Dickson algebra with divisors of zero. Hence $|D_1| \neq 0$.

THEOREM 3. *The order of the first liberal of an alternative algebra $\mathfrak{A}$ is equal to the nullity of the discriminant matrices of $\mathfrak{A}$.*

Since D_1 is a symmetric matrix by Lemma 1, a nonsingular M can be chosen so that $MD_1\overline{M}$ is a matrix of type (4) with a nonsingular matrix in the upper left-hand corner. It is then evident that the order of $\mathfrak{T}_1$ is equal to the nullity of D_1.

The following corollary then follows immediately from Lemma 2.

COROLLARY 3.1. *The first liberal of $\mathfrak{A}$ is zero if and only if $\mathfrak{A}$ is separable.*

Hence we have the corollary.

COROLLARY 3.2. *If* $\mathfrak{A}$ *is simple, then* $\mathfrak{T}_1=\mathfrak{A}$ *if and only if* $\mathfrak{A}$ *is not separable.*

And since a semi-simple algebra is a direct sum of simple algebras we obtain the corollary.

COROLLARY 3.3. *If* $\mathfrak{A}$ *is semi-simple, the first liberal is the direct sum of all inseparable components of* $\mathfrak{A}$.

For the basis $e_1, e_2, \cdots, e_n$ of $\mathfrak{A}$, let $e_{m+1}, \cdots, e_n$ be a basis for $\mathfrak{T}_1$. It then follows from Theorem 3 and the fact that $e_1, e_2, \cdots, e_n$ is also a basis for $\mathfrak{A}_{\mathfrak{K}}$ that $(\mathfrak{T}_1)_{\mathfrak{K}}$ is the first liberal of $\mathfrak{A}_{\mathfrak{K}}$.

THEOREM 4. *If* $\mathfrak{T}_1$ *is the first liberal of an alternative algebra* $\mathfrak{A}$, *then* $\mathfrak{A}/\mathfrak{T}_1$ *is separable.*

Since $\mathfrak{T}_1\supseteq\mathfrak{N}$, $\mathfrak{A}/\mathfrak{T}_1$ is semi-simple. If $\mathfrak{K}$ is any extension of $\mathfrak{F}$, then $(\mathfrak{A}/\mathfrak{T}_1)_{\mathfrak{K}}=\mathfrak{A}_{\mathfrak{K}}/(\mathfrak{T}_1)_{\mathfrak{K}}$ is semi-simple for the same reason and hence $\mathfrak{A}/\mathfrak{T}_1$ is separable.

3. **The liberal.** Before defining the *liberal* we first prove a useful lemma.

LEMMA 3. *If* $\mathfrak{B}$ *is an ideal of an alternative algebra* $\mathfrak{A}$ *such that* $\mathfrak{A}/\mathfrak{B}$ *is separable, then there is an algebra* $\mathfrak{S}$ *such that*

$$\mathfrak{A} = \mathfrak{B} + \mathfrak{S}, \qquad \mathfrak{S} \cong \mathfrak{A}/\mathfrak{B}.$$

If $\mathfrak{N}$ is the radical of $\mathfrak{A}$, then $\mathfrak{B}\supseteq\mathfrak{N}$. Let $\mathfrak{A}_0=\mathfrak{A}/\mathfrak{N}$, $\mathfrak{B}_0=\mathfrak{B}/\mathfrak{N}$ so that $\mathfrak{B}_0$ is an ideal of $\mathfrak{A}_0$. Since $\mathfrak{A}_0$ is semi-simple so is $\mathfrak{B}_0$, and $\mathfrak{A}_0=\mathfrak{B}_0\oplus\mathfrak{C}_0$ where $\mathfrak{C}_0\cong\mathfrak{A}_0/\mathfrak{B}_0$. But $\mathfrak{A}_0/\mathfrak{B}_0\cong\mathfrak{A}/\mathfrak{B}$ so $\mathfrak{C}_0$ is separable. Let $\mathfrak{C}$ be the sub-algebra of $\mathfrak{A}$ such that $\mathfrak{C}/\mathfrak{N}=\mathfrak{C}_0$. Then $\mathfrak{N}$ is the radical of $\mathfrak{C}$ and by Theorem 1 there is an algebra $\mathfrak{S}$ such that $\mathfrak{C}=\mathfrak{S}+\mathfrak{N}$, $\mathfrak{S}\cong\mathfrak{C}_0$. Now the intersection $\mathfrak{D}$ of $\mathfrak{B}$ and $\mathfrak{S}$ is an ideal of $\mathfrak{S}$ and hence is semi-simple. But the intersection of $\mathfrak{B}_0$ and $\mathfrak{C}_0$ is zero, so $\mathfrak{D}\subseteq\mathfrak{N}$. Hence $\mathfrak{D}=0$ and the order of $\mathfrak{B}$ plus the order of $\mathfrak{S}$ is the order of $\mathfrak{A}$ so that $\mathfrak{A}$ is the supplementary sum $\mathfrak{B}+\mathfrak{S}$ where $\mathfrak{S}\cong\mathfrak{A}/\mathfrak{B}$.

Let $\mathfrak{N}$ be the radical of an arbitrary alternative algebra $\mathfrak{A}$ and let $\mathfrak{T}'$ be the first liberal of the semi-simple algebra $\mathfrak{A}/\mathfrak{N}$. Then by Corollary 3.3, $\mathfrak{T}'$ is the direct sum of all inseparable components of $\mathfrak{A}/\mathfrak{N}$, and we have $\mathfrak{A}/\mathfrak{N}=\mathfrak{T}'\oplus\mathfrak{S}'$ where $\mathfrak{S}'$ is the direct sum of all separable components of $\mathfrak{A}/\mathfrak{N}$. The unique ideal $\mathfrak{T}\supseteq\mathfrak{N}$ of $\mathfrak{A}$ such that $\mathfrak{T}/\mathfrak{N}=\mathfrak{T}'$ is then the minimum ideal of $\mathfrak{A}$ such that $\mathfrak{A}/\mathfrak{T}$ is separable. We shall call $\mathfrak{T}$ the *liberal* of $\mathfrak{A}$. From this definition we see that the liberal reduces to the radical if and only if $\mathfrak{A}/\mathfrak{N}$ is separable. Also from Theorem 4 we conclude that $\mathfrak{T}\subseteq\mathfrak{T}_1$ and hence $T(x)=0$ for all x in $\mathfrak{T}$.

By Lemma 3 we have the following theorem.

THEOREM 5. *If $\mathfrak{L}$ is the liberal of an arbitrary alternative algebra $\mathfrak{A}$, then there is an algebra $\mathfrak{S}$ such that*

$$\mathfrak{A} = \mathfrak{S} + \mathfrak{L}, \qquad \mathfrak{S} \cong \mathfrak{A}/\mathfrak{L}. \tag{5}$$

Since $\mathfrak{L} = \mathfrak{N}$ when $\mathfrak{A}/\mathfrak{N}$ is separable, the decomposition (5) is the same as (1) when Theorem 1 holds. Moreover there is always a decomposition (5) even when there is no decomposition (1). This decomposition (5) may be trivial of course, that is, when $\mathfrak{L} = \mathfrak{A}$, but then there is no proper ideal $\mathfrak{J}$ of $\mathfrak{A}$ such that $\mathfrak{A}/\mathfrak{J}$ is separable. Examples can easily be constructed in which there are nontrivial decompositions (5) and no decomposition (1). It is also true that the liberal and the first liberal do not always coincide.

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