AVERAGES OF THE COEFFICIENTS OF SCHLICHT FUNCTIONS

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We shall consider throughout this paper a function

\[ f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_1 = 1, \]

analytic and schlicht in the unit circle. According to a classical conjecture of Bieberbach, \(|a_n| \leq n\). Recently Bazilevic has proved that \(\lim \sup_{n \to \infty} \frac{|a_n|}{n} \leq e/2\) [1]². Our interest lies in the average behavior of the coefficients. It is clear that if the conjecture holds, then

\[ \left| \sum_{j=1}^{n} a_j \right| / C_{n+1,2} \leq 1. \]

More generally, let us define

\[ S_n(k) = \sum_{j=0}^{n-1} C_{j+k-1, k-1} a_{n-j} \quad (k \geq 1) \]

and

\[ \sigma_n(k) = \left| S_n(k) \right| / C_{n+k,k+1}. \]

If \(|a_n| \leq n\), then

\[ \left| S_n(k) \right| \leq \sum_{j=0}^{n-1} C_{j+k-1,k-1}(n-j) = C_{n+k,k+1} \]

so that \(\sigma_n(k) \leq 1\). It is easy to see that the result of Bazilevic implies that

\[ \lim_{n \to \infty} \sigma_n(k) \leq \frac{e}{2}. \]

We prove two theorems concerning the averages \(\sigma_n(k)\). Using only classical results we obtain a bound on \(\lim \sup_{n \to \infty} \sigma_n(k)\) and show that this bound tends to unity for large \(k\). By applying recent information concerning the map of the circle \(|z| = r < 1\) by the function \(f(z)\), we get estimates on \(\lim \sup_{n \to \infty} \sigma_n(k)\) for small \(k\).

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² Numbers in brackets refer to the bibliography at the end of the paper.
Theorem 1. Let $k > 1$. Then
\[
\limsup_{n \to \infty} \sigma_n(k) \leq \frac{e^{k+1} \Gamma(k+2) \Gamma(k-1)}{(k+1)k^{k+1/2}\Gamma(k/2)} = A(k),
\]
and $\lim_{k \to \infty} A(k) = 1$.

Proof. We write
\[
S_n(k) = \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)}{z^{n+1}(1-z)^b} \, dz
\]
(1)
Hence
\[
|S_n(k)| \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(re^{i\theta})|}{|1-re^{i\theta}|^b} \, d\theta
\]
By the well known "Distortion Theorem"
\[
\max_{0 \leq \theta < 2\pi} \left| f(re^{i\theta}) \right| \leq \frac{r}{(1-r)^2} \quad (r < 1).
\]
To estimate the integral expression we write
\[
|1-re^{i\theta}| = (1-2r \cos \theta + r^2)^{1/2},
\]
so that
\[
\frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{|1-re^{i\theta}|^b} = \frac{1}{2\pi(1-r^2)^{b/2}} \int_0^{2\pi} \frac{(1-r^2)^{b/2}}{(1-2r \cos \theta + r^2)^{b/2}} d\theta
\]
\[
= (1-r^2)^{-b/2} P_{k/2-1} \left( \frac{1+r^2}{1-r^2} \right).
\]
Here $P_{n(x)}$ is the Legendre function of the first kind of order $n$. Since $\lim_{x \to \infty} P_n(x)/x^n = 2^{-n}\Gamma(2n+1)/\Gamma^2(n+1)$ [3, p. 62], we may write
\[
|S_n(k)| \leq r^{-n+1}(1-r)^{-2}(1-r^2)^{-b/2} \phi_k(r),
\]
where $\lim_{r \to 1} \phi_k(r) = 2^{-b+1/2} \Gamma(k-1)/\Gamma^2(k/2)$.
Thus far $r$ has been any number between 0 and 1. We now specify
$r = 1 - (k+1)/n$. Then
\[
\left| \sigma_n(k) \right| \leq \left( 1 - \frac{k + 1}{n} \right)^{-n+1} \left( \frac{k + 1}{n} \right)^{-k-1} (1 + r)^{-k/2} 
\cdot \left( \frac{1 + r^2}{1 + r} \right)^{k/2 - 1} \phi_k(r) (C_{n+k,k+1})^{-1}.
\]

Since $C_{n+k,k+1} \leq n^{k+1}/\Gamma(k+2)$, we readily compute $\lim \sup_{n \to \infty} \sigma_n(k) \leq A(k)$, where
\[
A(k) = \frac{e^{k+1} \Gamma(k + 2) \Gamma(k - 1)}{(k + 1)^{k+1} 2^{k-1} \Gamma^2(k/2)}.
\]

That $\lim_{k \to \infty} A(k) = 1$ may now be verified by using Stirling’s formula for $\Gamma(k)$.

While the numbers $A(k)$ do tend to unity they decrease very slowly. Computations yield $A(2) = 2.23$, $A(4) = 1.42$, $A(6) = 1.26$, $A(10) = 1.15$, $A(20) = 1.07$. Hence even $A(4)$ is greater than $e/2$. A better estimate of $\lim \sup_{n \to \infty} \sigma_n(k)$ for small $k$ can be obtained by use of the following lemma, recently announced by Bazilevič [1].

**Lemma.** The intersection of the circumference $|w| = x, x \geq r e^{\pi i} / 2$, with the domain $D(r)$ on which $f(z)$ maps $|z| \leq r < 1$ has linear measure not greater than that of the intersection of the same circumference with the domain $D^*(r)$ on which $f^*(z) = z/(1-z)^2$ maps $|z| \leq r$.

It follows at once from the lemma that the area $\psi(r)$ of the region $D(r)$ is not greater than $\pi r^2 e^{2 \pi i / 2}$ plus the area $\psi^*(r)$ of $D^*(r)$. Further
\[
\psi^*(r) = \int_0^{2\pi} d\theta \int_0^r r |f^*(re^{i\theta})|^2 dr = \pi \sum_{i=1}^{\infty} j^3 r^{2i} = \frac{\pi r^2 (1 + 4r^2 + r^4)}{(1 - r^2)^4}.
\]

We may now prove our second theorem.

**Theorem 2.** Let $k \geq 1$. Then
\[
\lim \sup_{n \to \infty} \sigma_n(k) \leq \frac{ke^{k+1} \Gamma^{1/2}(2k - 1)}{(k + 1)^{k+1} 2^{k+1/2}} = B(k).
\]
In particular,

\[ B(1) = 1.307, \quad B(2) = 1.116, \quad B(3) = 1.109. \]

**Proof.** We apply Schwarz's inequality to (1) to get

\[
|S_n(k)|^2 \leq r^{-2n} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \right\} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{1 - re^{i\theta}} \right\}.
\]

To estimate the first integral we write

\[
I = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta
\]

\[
= \sum_{j=1}^{\infty} a_j^2 r^{2j}
\]

\[
= 2 \int_0^r \sum_{j=1}^{\infty} j a_j^2 i^{-j} dr
\]

\[
= \frac{2}{\pi} \int_0^r \frac{\psi(r)}{r} dr.
\]

Hence (2) yields

\[
I \leq \frac{2}{\pi} \int_0^r \left\{ r \pi r e^{r/e} + \frac{\pi r (1 + 4r^2 + r^4)}{(1 - r^2)^4} \right\} dr.
\]

An integration by parts then gives

\[
I \leq \frac{2}{(1 - r)^3} + \frac{g(r)}{(1 - r)^2},
\]

where \( g(r) \) is a function bounded for \( 0 \leq r \leq 1 \).

The second integral of (4) can be handled as in Theorem 1. Thus

\[
|S_n(k)|^2 \leq r^{-2n} \left\{ \frac{2}{(1 - r)^3} + \frac{g(r)}{(1 - r)^2} \right\} (1 - r^2)^{-k} \left\{ \frac{1 + r^2}{1 - r} \right\}.
\]

On choosing

\[
r = 1 - \frac{k + 1}{n}
\]

and carrying out the computations as before, we get the results asserted.

It is interesting to note that for values of \( k > 3 \) the numbers \( B(k) \)
defined in (3) increase, behaving like $2^{-1} \left\{ \pi k \right\}^{1/4}$ for large $k$. The technique of using the Schwarz inequality is thus ineffective for the study of $\lim \sup_{n \to \infty} \sigma_n(k)$ for all but the smaller values of $k$.

BIBLIOGRAPHY


UNIVERSITY OF CALIFORNIA, LOS ANGELES