The purpose of this note is to give a simplified proof of a theorem of Gelfand and Šilov in the theory of normed, commutative rings. Let $K$ be a complex Banach space which is also a commutative ring with unit element $e$, the norm being subject to the conditions $\|e\| = 1$ and $\|xy\| \leq \|x\| \|y\|$ for all $x$ and $y$ in $K$. Let $\mathcal{M}$ be the space of maximal ideals of $K$. Then, for each element $x \in K$, and each maximal ideal $M \in \mathcal{M}$, there is a unique complex number $x(M)$ defined by

$$x = x(M)e \pmod{M}$$

and having the following properties:

1. $e(M) = 1$,
2. $(x + y)(M) = x(M) + y(M)$,
3. $(xy)(M) = x(M)y(M)$,
4. $(ax)(M) = ax(M)$,
5. $|x(M)| = \|x\|$,

for all $x$ and $y$ in $K$ and any complex number $\alpha$ [1].

Gelfand [1] introduces a topology in $\mathcal{M}$ by defining a neighborhood $U$ of $M_0$ as follows:

$$U = \{ M ; \ |x_i(M) - x_i(M_0)| < \alpha_i ; x_i \in K ; \alpha_i > 0 ; i = 1, 2, \ldots, k \} ;$$

and he proves that in this topology, $\mathcal{M}$ is a compact Hausdorff space, and that this is the unique topology in which all functions $x(M)$, $x \in K$, are continuous and $\mathcal{M}$ is compact.

If the ring $K$ also has the property that, for every $x \in K$, there exists an $x^* \in K$ such that $x(M)$ and $x^*(M)$ are complex conjugates for all $M$, then the functions $x(M)$ are dense in the set of all continuous functions on $\mathcal{M}$. This result is proved by Gelfand and Šilov [2] by a method depending on two other topologies in the space $\mathcal{M}$. We give here a simpler and more direct proof of this theorem, making use of only the one topology defined above.

**Lemma.** If $F_1$ and $F_2$ are any two disjoint closed sets in $\mathcal{M}$, and $0 < \epsilon_1 < 1$, $0 < \epsilon_2 < 1$, there exists an $x \in K$ such that

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Numbers in brackets refer to the references at the end of the paper.
Proof. Let \( M_0 \in F_1 \). Let \( U = \{ M ; |x_i(M) - x_i(M_0)| < \alpha ; \ i = 1, 2, \ldots, k \} \) be a neighborhood of \( M_0 \) which does not intersect \( F_2 \). Let \( y' = \sum_{i=1}^{k} x_i(x_i')^* \). Then \( y'(M) \) is non-negative for all \( M \), is zero for \( M = M_0 \), and has a positive lower bound \( \alpha^2 \) for \( M \in F_2 \). Let \( y = y'/\|y'\|, \ 2\delta = \alpha^2/\|y'\| \leq 1 \); then

\[
0 \leq y(M) \leq 1, \quad M \in \mathcal{M},
\]

\[
y(M_0) = 0,
\]

\[
0 < 2\delta \leq y(M) \leq 1, \quad M \in F_2.
\]

With each \( M_0 \in F_1 \) we associate a \( y \) with the properties just given and a neighborhood of \( M_0 \): \( U = \{ M ; y(M) < \delta \} \). From these neighborhoods we select a finite covering of \( F_1 \), denoted by \( U_1, U_2, \ldots, U_N \), and we let the \( y \) and \( \delta \) associated with \( U_k \) be denoted by \( y_k \) and \( \delta_k \). Now let \( z_k = e^{-(e - y_k)}^m \), where \( n \) and \( m \) are positive integers. We then have

\[
0 \leq z_k(M) \leq 1, \quad M \in \mathcal{M},
\]

\[
0 \leq z_k(M) < 1 - (1 - \delta_k)^m, \quad M \in U_k,
\]

\[
1 - (1 - 2\delta)^m \leq z_k(M) \leq 1, \quad M \in F_2.
\]

If, for each value of \( n \), we now choose \( m \) as the integer nearest \( (2/3\delta_k)^n \), then, as \( n \to \infty \), \( m \log (1 - \delta_k^m) \to 0 \) and \( m \log (1 - 2\delta_k^m) \to \infty \); hence \( (1 - \delta_k)^m \to 1 \) and \( (1 - 2\delta_k^m) \to 0 \). This follows from the inequality \( h \leq \log (1 - h) \leq 2h \) for \( 0 < h \leq 1/2 \). We can therefore choose \( n \) so large that

\[
0 \leq z_k(M) < \epsilon_k, \quad M \in U_k,
\]

\[
(1 - \epsilon_k)^{1/N} \leq z_k(M) \leq 1, \quad M \in F_2.
\]

The function \( x = z_1z_2 \cdots z_N \) will then satisfy the conditions of the lemma.

Theorem. For any complex-valued function \( f(M) \), continuous on \( \mathcal{M} \), and any \( \epsilon > 0 \), there exists an \( x \in \mathcal{K} \) such that \( |f(M) - x(M)| < \epsilon \) for all \( M \).

Proof. We first prove the theorem for a real-valued continuous function \( f(M) \). Let \( K_R \) be the set \( \{ x \} \) where \( x \in \mathcal{K} \) and \( x(M) \) is real-valued, and let
\[ a = \inf_{z \in K_R} \sup_{M \in \mathcal{M}} \left| f(M) - x(M) \right|. \]

Let us assume \( a > 0 \). Then there exists an \( x_0 \in K_R \) such that

\[ \sup_{M \in \mathcal{M}} \left| f(M) - x_0(M) \right| < 1.1a. \]

Let \( g(M) = f(M) - x_0(M) \), and let

\[ F_1 = \{ M ; g(M) \geq .5a \}, \]
\[ G = \{ M ; \left| g(M) \right| < .5a \}, \]
\[ F_2 = \{ M ; g(M) \leq -.5a \}. \]

By the preceding lemma we can find \( x_1 \in K_R, x_2 \in K_R \) such that

\[ 0 \leq x_1(M) \leq .4a, \quad 0 \leq x_2(M) \leq .4a, \quad M \in \mathcal{M}, \]
\[ 0 \leq x_1(M) \leq .1a, \quad .3a \leq x_2(M) \leq .4a, \quad M \in F_2, \]
\[ .3a \leq x_1(M) \leq .4a, \quad 0 \leq x_2(M) \leq .1a, \quad M \in F_1. \]

If \( x = x_0 + x_1 - x_2 \), then \( x \in K_R \) and

\[ \left| f(M) - x(M) \right| < .9a, \quad \text{for all } M \in \mathcal{M}. \]

This is a contradiction; hence \( a = 0 \) for any real-valued continuous \( f(M) \).

If \( f(M) \) is complex-valued, we can apply the result just proved to the real and imaginary parts of \( f(M) \) separately.

**References**


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