A GENERALIZATION OF LAPLACE'S METHOD[1]

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1. **Introduction.** In many instances one is concerned with the asymptotic behavior of a function as two or more parameters become large simultaneously. One example is that of Bessel functions of large order and argument. Another is the case of the incomplete gamma function $\Gamma(a+1, x)$ where both a and x become large. If one sets both a and x proportional to a large parameter the problem is reduced to the simpler case of a single parameter which can be investigated quite easily by, for example, Laplace's method. (This reduction is also made in the case of the Bessel function $J_\nu(z)$ which may then be treated by the method of steepest descents.) However useful this may be, it throws little light on the character of the function when a and x are in some sense nearly equal as they grow large, say when $a-x=o(x)$. Thus it seems worth while to investigate functions in which two parameters are not rigidly bound together, but are allowed more relative freedom.

For functions involving only one large parameter Laplace's method has proved to be a valuable tool. One considers an integral of the form

$$I = \int_a^b f(t) \exp\left[-k\phi(t)\right]dt$$

where $\phi(t)$ has a flat minimum at some interior point t_0 ($\phi'(t_0)=0$, $\phi''(t_0)>0$) as $k\to+\infty$. One assumes ϕ has only one such point, for if it had a finite number the integral can be reduced to a finite number of integrals each such that ϕ has only one minimum in the range of integration. Under reasonably relaxed conditions one shows[3] that

$$I \sim f(t_0) \exp\left[-k\phi(t_0)\right]\left(\frac{2\pi}{k\phi''(t_0)}\right)^{1/2} \qquad \text{as } k\to+\infty .$$

In the present paper we shall investigate a generalization of Laplace's method where we consider integrals of the form

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[3] See, for example, D. V. Widder, *The Laplace transform*, Princeton, 1946, pp. 277–280.

$$I_{h,k} = \int_a^b f(t) \exp\left[-h\phi(t) + k\psi(t)\right]dt$$

where h and k both grow but h faster than k, say. In which case the neighborhoods of the minimal points of ϕ could be expected to give the most important contributions. Again we shall assume we have only one minimal point. We shall simplify even further by splitting the integral at the minimum. Hence we shall assume the single minimum occurs at one end, say the lower, which we take to be the origin. By dividing out constant factors we can normalize to $\phi(0)=\psi(0)=0$. We shall need to expand our functions near the origin. All of the requirements will be covered by the following general assumptions, which will hereafter be referred to as assumptions A:

A_1. $\phi(0)=\phi'(0)=0$, $\phi''(0)>0$, $\phi(t)\in C^3$ in I: $-\eta<t<\eta\leqq b$ *and* $\phi(t)$ *is positive and nondecreasing in* $0<t\leqq b$;

A_2. $\psi(0)=0$, $\psi(t)\in C^2$ *in* I, $\psi(t)$ *is real and continuous in* $0\leqq t\leqq b$;

A_3. $f(t)\in L$, $0\leqq t\leqq b$, $f(t)$ *continuous at* $t=0$ *and* $f(0)\neq 0$;

A_4. h, $k\to+\infty$, $k=o(h)$.

Some of these conditions could be relaxed. For example, $\psi(t)$ need not be continuous in the whole interval: we need the existence of the integral and effective boundedness of ψ. Condition A_2 gives us both. Further, if $b=+\infty$, A_3 may be relaxed to $f(t)\in L$ for every finite subinterval and existence of the integral for sufficiently large h and k.

2. **Statement of results.** Throughout the remainder of the paper we shall use

$$A \sim B \quad \text{to mean} \quad \lim \frac{A}{B} = 1,$$

and all limits will mean the limit as both h and k tend to $+\infty$ in the prescribed manner. A similar remark applies to order symbols.

Under the assumptions A we shall consider the integral

$$I_{h,k} = \int_0^b f(t) \exp\left[-h\phi(t) + k\psi(t)\right]dt$$

and prove the results stated below. First we consider cases under which the behavior of $h\phi(t)$ at the origin completely determines the integral.

THEOREM 1. *If either* $k=o(h^{1/2})$ *or* $\psi'(0)=0$, *then*

$$I_{h,k} \sim \frac{1}{2} f(0) \left(\frac{2\pi}{h\phi''(0)}\right)^{1/2}.$$

Next are the two cases in which the behavior of $k\psi(t)$ at the origin plays a significant role:

THEOREM 2. *If* $0<\liminf k/h^{1/2}$ *and* $\limsup k/h^{1/2}<\infty$, *then*

$$I_{h,k} \sim \frac{1}{2} f(0) \left(\frac{2\pi}{h\phi''(0)}\right)^{1/2} \cdot \exp\left(\frac{\psi'^2(0)k^2}{2\phi''(0)h}\right)\left[1+\frac{2}{\pi^{1/2}}\operatorname{Erf}\frac{\psi'(0)k}{(2h\phi''(0))^{1/2}}\right]$$

(*here* $\operatorname{Erf} z=\int_0^z e^{-u^2}du$).

THEOREM 3. *If* $h^{1/2}=o(k)$ *and* $\psi'(0)<0$, *then*

$$I_{h,k} \sim -\frac{f(0)}{k\psi'(0)}.$$

And finally we have the case in which the behavior of ϕ and ψ near, not only at, the origin becomes important.

THEOREM 4. *If* $h^{1/2}=o(k)$ *and* $\psi'(0)>0$, *then*

$$I_{h,k} \sim f(0)\left(\frac{2\pi}{h\phi''(0)}\right)^{1/2} \exp\left[-h\phi(\tau)+k\psi(\tau)\right]$$

where τ *is defined by* $h\phi'(\tau)=k\psi'(\tau)$.

The last theorem is in terms of the quantity τ which depends, to a very great extent, upon the precise relation between h and k. If one makes more stringent assumptions the result can be given a more explicit form. For example we state the following corollary.

COROLLARY. *If* $\psi'(0)>0$, $h^{1/2}=o(k)$, *and* $k=o(h^{2/3})$, *then*

$$I_{h,k} \sim f(0)\left(\frac{2\pi}{h\phi''(0)}\right)^{1/2} \exp\left(\frac{\psi'^2(0)k^2}{2\phi''(0)h}\right).$$

Clearly, if the minimal point for ϕ occurs in the interior of the range of integration (which point can be taken as origin) one gets double the result in the case of Theorem 1. If, however, $h^{1/2}=o(k)$ and $\psi'(0)\neq 0$, the integral can be reduced to two integrals of the form $I_{h,k}$. In one of these $\psi'(0)>0$ and in the other $\psi'(0)<0$. The contribution from the one with positive $\psi'(0)$ obviously dominates. In the case of Theorem 2, one doubles the result and sets the error function equal to zero, for in the addition of the two integrals the error functions from the two sides have opposite signs.

3. **Proofs.** We first want to estimate the value of τ defined by $h\phi'(\tau)=k\psi'(\tau)$. This equation can be written

$$h\phi''(0)\tau + h\frac{\phi'''(t_1)}{2}\tau^2 = k\psi'(0) + k\psi''(t_2)\tau$$

where t_1 and t_2 are between τ and the origin. Solving this quadratic equation in τ, and using the boundedness of $\phi'''(t_1)$ and $\psi''(t_2)$, we see that

$$\tau = \frac{\psi'(0)k}{\phi''(0)h} + O\left(\frac{k^2}{h^2}\right).$$

In particular, as is easily seen from the definition, τ is zero if $\psi'(0)=0$.

We shall prove the theorems under the assumption that $f(t)\equiv 1$, $0\leqq t\leqq b$, then later remove this. To this end we consider the integral

$$\begin{aligned} I_1 &= \int_0^b \exp\left[-h(\phi(t)-\phi(\tau)) + k(\psi(t)-\psi(\tau))\right]dt \\ &= \int_0^c + \int_c^b = I_2 + I_3 \end{aligned}$$

where, given a sufficiently small $\epsilon>0$, we choose c so small that $0<\phi''(0)-\epsilon/2\leqq\phi''(t)\leqq\phi''(0)+\epsilon/2$ for $0\leqq t\leqq c$. Let M be max $|\psi(t)|$, $0\leqq t\leqq b$. Using the fact that $\tau=o(1)$ we see that, for sufficiently large h and k,

$$I_3 \leqq (b-c)\exp\left[-h\phi(c) + h\phi(\tau) + 2kM\right] \leqq \exp\left[-h\phi(c)/2\right].$$

If we expand $\phi(t)$ and $\psi(t)$ about $t=\tau$ and recall that $h\phi'(\tau)=k\psi'(\tau)$, we obtain

$$I_2 = \int_0^c \exp\left[-\frac{1}{2}h\left(\phi''(t_1) - \frac{k}{h}\psi''(t_2)\right)(t-\tau)^2\right]dt$$

where t_1 and t_2 are between t and τ. We now break this up into the sum of the integrals from 0 to τ, and from τ to c. In the first we substitute $t-\tau=-y/h^{1/2}$, and in the second $t-\tau=y/h^{1/2}$. This gives us

$$\text{(B)}\qquad \begin{aligned} I_2 = \frac{1}{h^{1/2}}\int_0^{\tau h^{1/2}} &\exp\left[-\left(\phi''(t_1) - \frac{k}{h}\psi''(t_2)\right)\frac{1}{2}y^2\right]dy \\ &+ \frac{1}{h^{1/2}}\int_0^{(c-\tau)h^{1/2}} \exp\left[-\frac{1}{2}\left(\phi''(t_1) - \frac{k}{h}\psi''(t_2)\right)y^2\right]dy. \end{aligned}$$

It is clear from equation (B) that we must distinguish three cases

according as the range of integration of the first integral becomes zero, stays finite, or becomes infinite. We first suppose that $k=o(h^{1/2})$ so that $\tau h^{1/2}$ vanishes in the limit. For sufficiently large h and k the bracket in the exponential in the first integral is bounded above by $2M'$, say, and below by $m>0$, and we have from (B) that

$$\frac{1}{h^{1/2}}\int_0^{(c-\tau)h^{1/2}} \exp\left[-\frac{1}{2}(\phi''(0)+\epsilon)y^2\right]dy + \tau e^{-M'\tau^2 h} \leqq I_2$$

$$\leqq \tau + \frac{1}{h^{1/2}}\int_0^{(c-\tau)h^{1/2}} \exp\left[-\frac{1}{2}(\phi''(0)-\epsilon)y^2\right]dy.$$

We multiply this inequality by $h^{1/2}$ and let $h\to\infty$:

$$\frac{1}{2}\left(\frac{2\pi}{\phi''(0)+\epsilon}\right)^{1/2} \leqq \liminf h^{1/2}I_2 = \liminf h^{1/2}I_1 \leqq \limsup h^{1/2}I_1$$

$$= \limsup h^{1/2}I_2 \leqq \frac{1}{2}\left(\frac{2\pi}{\phi''(0)-\epsilon}\right)^{1/2}.$$

But I_1 is independent of ϵ. Thus if we let $\epsilon\to 0$, we have

$$\lim h^{1/2}I_1 = \frac{1}{2}\left(\frac{2\pi}{\phi''(0)}\right)^{1/2},$$

from which

$$I_{h,k} \sim \frac{1}{2}\left(\frac{2\pi}{h\phi''(0)}\right)^{1/2} \exp\left[-h\phi(\tau)+k\psi(\tau)\right],$$

and by expanding $-h\phi(\tau)+k\psi(\tau)$ about the origin we see that

$$\exp\left[-h\phi(\tau)+k\psi(\tau)\right] \sim 1.$$

Since for $\psi'(0)=0$ we have $\tau=0$ for all h, k, the above estimates are valid if we have merely $k=o(h)$ and $\psi'(0)=0$. This establishes Theorem 1 for the special case $f\equiv 1$.

Now we turn to Theorem 2 in which, for sufficiently large h and k, $k/h^{1/2}$ is bounded away from both 0 and ∞. Let us consider the quantity

$$J_1 = h^{1/2}I_2 - \int_0^{\psi'(0)k/\phi''(0)h^{1/2}} \exp\left[-\frac{1}{2}\phi''(0)y^2\right]dy$$

$$= h^{1/2}I_2 - \left(\frac{2}{\phi''(0)}\right)^{1/2} \operatorname{Erf}\frac{\psi'(0)k}{(2h\phi''(0))^{1/2}}.$$

From(B) we have that

$$J_1 = \int_0^{\tau h^{1/2}} \exp\left[-\frac{1}{2}\left(\phi''(t_1) - \frac{k}{h}\psi''(t_2)\right)y^2\right] dy$$
$$- \int_0^{\psi'(0)k/\phi''(0)h^{1/2}} \exp\left[-\frac{1}{2}\phi''(0)y^2\right] dy$$
$$+ \int_0^{(c-\tau)h^{1/2}} \exp\left[-\frac{1}{2}\left(\phi''(t_1) - \frac{k}{h}\psi''(t_2)\right)y^2\right] dy.$$

Denoting by D the difference of the first two integrals, we have

$$D = \int_0^{\psi'(0)k/\phi''(0)h^{1/2}} \exp\left[-\frac{1}{2}\phi''(0)y^2\right]$$
$$\cdot \left\{\exp\left[-\frac{1}{2}\left(\phi''(t_1) - \phi''(0) - \frac{k}{h}\psi''(t_2)\right)y^2\right] - 1\right\} dy$$
$$+ \int_{\psi'(0)k/\phi''(0)h^{1/2}}^{\psi'(0)k/\phi''(0)h^{1/2}} + O(1/h^{1/2}) \exp\left[-\frac{1}{2}\left(\phi''(t_1) - \frac{k}{h}\psi''(t_2)\right)y^2\right] dy.$$

The second integral vanishes as $h \to \infty$ since the integrand is bounded and the range of integration vanishes. Thus

$$D \leqq K\left[\max_{0 \leqq t_1 \leqq \tau} |\phi''(t_1) - \phi''(0)| + \frac{k}{h}\max_{0 \leqq t_2 \leqq \tau} |\psi''(t_2)|\right]$$
$$\times \int_0^{\psi'(0)k/\phi''(0)h^{1/2}} \exp\left\{-\frac{1}{2}\phi''(0)y^2\right\} y^2 dy + o(1)$$

where K is a constant. Clearly the bracket [ ] vanishes as $h \to +\infty$ and the integral is finite. Hence $D = o(1)$.

If we let $h \to +\infty$ we then have

$$\frac{1}{2}\left(\frac{2\pi}{\phi''(0) + \epsilon}\right)^{1/2} \leqq \liminf J_1$$
$$= \liminf \left[h^{1/2}I_1 - \left(\frac{2}{\phi''(0)}\right)^{1/2} \operatorname{Erf} \frac{\psi'(0)k}{(2h\phi''(0))^{1/2}}\right]$$
$$\leqq \limsup \left[h^{1/2}I_1 - \left(\frac{2}{\phi''(0)}\right)^{1/2} \operatorname{Erf} \frac{\psi'(0)k}{(2h\phi''(0))^{1/2}}\right]$$
$$= \limsup J_1 \leqq \frac{1}{2}\left(\frac{2\pi}{\phi''(0) - \epsilon}\right)^{1/2}.$$

Again let $\epsilon \to 0$, and we can estimate

$$\exp\left[-h\phi(\tau) + k\psi(\tau)\right] \sim \exp\left[\frac{\psi'^2(0)k^2}{2\phi''(0)h}\right]$$

to obtain

$$I_{h,k} \sim \frac{1}{2}\left(\frac{2\pi}{\phi''(0)h}\right)^{1/2} \exp\left[\frac{\psi'^2(0)k^2}{2h\phi''(0)}\right]\left[1 + \frac{2}{\pi^{1/2}} \operatorname{Erf} \frac{\psi'(0)k}{(2h\phi''(0))^{1/2}}\right],$$

which proves Theorem 2 for $f \equiv 1$.

We jump now to Theorem 4. We easily estimate from (B) that

$$\begin{aligned}
&\int_0^{\tau h^{1/2}} \exp\left[-\frac{1}{2}(\phi''(0) + \epsilon)y^2\right] dy \\
&\qquad + \int_0^{(c-\tau)h^{1/2}} \exp\left[-\frac{1}{2}(\phi''(0) + \epsilon)y^2\right] dy \leqq h^{1/2} I_2 \\
&\leqq \int_0^{\tau h^{1/2}} \exp\left[-\frac{1}{2}(\phi''(0) - \epsilon)y^2\right] dy \\
&\qquad + \int_0^{(c-\tau)h^{1/2}} \exp\left[-\frac{1}{2}(\phi''(0) - \epsilon)y^2\right] dy.
\end{aligned}$$

By the same method as before we let $h \to \infty$, $\epsilon \to 0$ and obtain the result

$$I_{h,k} \sim \left(\frac{2\pi}{h\phi''(0)}\right)^{1/2} \exp\left[-h\phi(\tau) + k\psi(\tau)\right]$$

and, under the additional assumption that $k = o(h^{2/3})$, we easily estimate the exponential by expanding the exponent around the origin and obtain

$$I_{h,k} \sim \left(\frac{2\pi}{h\phi''(0)}\right)^{1/2} \exp\left[\frac{\psi'^2(0)k^2}{2\phi''(0)h}\right].$$

This establishes both Theorem 4 and the corollary, for $f \equiv 1$.

We treat Theorem 3 by working directly with $I_{h,k}$. Let $\epsilon > 0$ be given, choose c as before and also so that $0 > \psi'(0) + \epsilon \geqq \psi'(t) \geqq \psi'(0) - \epsilon$, $0 \leqq t \leqq c$. Then break up the integral as before

$$I_{h,k} = \int_0^c + \int_c^b = I_4 + I_5.$$

Then

$$I_5 \leqq (b - c) \exp\left[-h\phi(c) + kM\right] \leqq \exp\left[-\frac{1}{2}h\phi(c)\right]$$

for h sufficiently large. And clearly

$$\text{(C)}\quad \begin{aligned} &\int_0^c \exp\left[-\frac{1}{2}h(\phi''(0)+\epsilon)t^2 + k(\psi'(0)-\epsilon)t\right]dt \\ &\qquad \leqq I_4 \leqq \int_0^c \exp\left[-\frac{1}{2}h(\phi''(0)-\epsilon)t^2 + k(\psi'(0)+\epsilon)t\right]dt. \end{aligned}$$

In order to estimate further we investigate the behavior of

$$J_2 = \int_0^c \exp\,[-h\gamma t^2 - k\delta t]dt$$

where γ, δ are positive constants. By completing the square and making an obvious change of variable we get

$$\begin{aligned} J_2 &= \frac{1}{(h\gamma)^{1/2}}\exp\left(\frac{k^2\delta^2}{4h\gamma}\right)\int_{k\delta/2(h\gamma)^{1/2}}^{c(h\gamma)^{1/2}+k\delta/2(h\gamma)^{1/2}} e^{-t^2}dt \\ &= \frac{1}{(h\gamma)^{1/2}}\exp\left(\frac{k^2\delta^2}{4h\gamma}\right)\left\{\int_0^{c(h\gamma)^{1/2}} - \int_0^{k\delta/2(h\gamma)^{1/2}} \right. \\ &\qquad\qquad \left. + \int_{c(h\gamma)^{1/2}}^{c(h\gamma)^{1/2}+k\delta/2(h\gamma)^{1/2}}\right\} e^{-t^2}dt. \end{aligned}$$

The last integral is clearly dominated by

$$\frac{k\delta}{2h\gamma}\exp\left\{\frac{k^2\delta^2}{4h\gamma} - c^2h\gamma\right\},$$

which vanishes exponentially. Estimating the other two integrals by the asymptotic expansion of the error function we obtain

$$J_2 = \frac{1}{k\delta} + \frac{1}{k}O\left(\frac{h}{k^2}\right).$$

Now inequality (C) becomes

$$\frac{1}{k}O\left(\frac{k}{k^2}\right) - \frac{1}{k}\frac{1}{\psi'(0)-\epsilon} \leqq I_4 \leqq -\frac{1}{k}\frac{1}{\psi'(0)+\epsilon} + \frac{1}{k}O\left(\frac{h}{k^2}\right).$$

We multiply by k, let $h\to\infty$, then let $\epsilon\to 0$ and we have finally

$$I_{h,k} \sim -\frac{1}{k\psi'(0)}.$$

It remains only to remove the restriction on $f(t)$. For Theorems 1, 2, 4 it is sufficient to show that

$$I_6 = h^{1/2}\int_0^b [f(t) - f(0)] \exp [-h(\phi(t) - \phi(\tau)) + k(\psi(t) - \psi(t))]dt = o(1).$$

Let $\epsilon > 0$ be given; choose c as before and with the additional restriction that $|f(t) - f(0)| \leqq \epsilon$, $0 \leqq t \leqq c$. Then

$$|I_6| \leqq \epsilon h^{1/2}\int_0^c \exp [-h(\phi(t)) - \phi(\tau)) + k(\psi(t) - \psi(\tau))]dt + o(1);$$

let $h \to \infty$:

$$|I_6| \leqq \epsilon \cdot \text{constant}.$$

Let $\epsilon \to 0$, and this establishes the result.

A similar estimate for Theorem 3 is easy:

$$J_3 = k\int_0^b \exp [-h\phi(t) + k\psi(t)] \, |f(t) - f(0)| \, dt$$

$$\leqq \epsilon k \int_0^c \exp [-h\phi(t) + k\psi(t)]dt + o(1).$$

Now let $k \to \infty$, $\epsilon \to 0$. This completes the proofs.

4. **An application.** We shall investigate the behavior of

$$\Gamma(a + 1, x) = \int_x^\infty e^{-u}u^a du, \qquad x, a > 0,$$

when x and a are "nearly" equal.[4] Set $u = x(t+1)$ and $a = x + y$ where $y = o(x)$ and get

$$\Gamma(x + y + 1, x) = e^{-x}x^{x+y+1}\int_0^\infty \exp [-x(t - \log (1 + t)) + y \log (1 + t)]dt.$$

By applying our theory to the integral we have immediately

(1) if $y = o(x^{1/2})$, then

$$\Gamma(x + y + 1, x) \sim \left(\frac{\pi}{2}\right)^{1/2} e^{-x}x^{x+y+1/2};$$

[4] Professor F. G. Tricomi has investigated the asymptotic character of the incomplete gamma function, and his results are to be published in the Mathematische Zeitschrift. In fact our interest in this problem arose from a reading of his paper. Our formulas (1) and (2) are equivalent to the first two terms of his formula (27).

(2) if $0<\liminf |y|/x^{1/2}$ and $\limsup |y|/x^{1/2}<\infty$, then

$$\Gamma(x+y+1,\ x)\sim\left(\frac{\pi}{2}\right)^{1/2}e^{y^2/2x-x}x^{x+y+1/2}\left[1+\frac{2}{\pi^{1/2}}\operatorname{Erf}\frac{y}{(2x)^{1/2}}\right];$$

(3) if $y>0$ and $x^{1/2}=o(y)$, then

$$\Gamma(x+y+1,\ x)\sim\left(\frac{x+y}{e}\right)^{x+y}(2\pi x)^{1/2};$$

(4) if $y<0$ and $x^{1/2}=o(|y|)$, then

$$\Gamma(x+y+1,\ x)\sim-\frac{1}{y}\,x^{x+y+1}e^{-x}.$$

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A RESIDUE THEOREM FOR FINITE BLASCHKE PRODUCTS

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1. Finite Blaschke products have long played a central role in the theory of bounded analytic functions, appearing frequently as functions enjoying various extremal properties. The present note is concerned with one such extremal property and its implications. Our principal result is:

If a finite Blaschke product has poles in the finite plane, then for at least one such pole the residue does not vanish.

Of course the presence of a simple pole in the finite plane renders the theorem trivial. However if one considers the case of a finite Blaschke product which is even and all of whose poles are multiple, then the existence of a nonzero residue is much more concealed. For the evenness of the Blaschke product would force the coefficient of z^{-1} in its expansion at ∞ to vanish so that the presence of residues of the desired type would escape detection on examination of the residue at ∞; on the other hand a direct study of the residues corresponding to the finite poles, while not out of the realm of possibility, leads in the case of a large number of poles to a computational morass. A further complicating factor is revealed by examples (§5) of finite Blaschke products with residue vanishing at some finite pole.

2. We recall that a finite Blaschke product is a rational function admitting a representation of the form

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