FRÉCHET AND KERÉKJÁRTÓ EQUIVALENCE

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1. Introduction. Let $P_1$, $P_2$ be two homeomorphic Peano spaces, let $P$ be a metric space with metric $d(p', p'')$, and let $T_1$, $T_2$ be continuous mappings from $P_1$, $P_2$ respectively into $P$. The reader is referred to Radó [1] for definitions and concepts used here.

$T_1$ and $T_2$ are called Fréchet equivalent, written for brevity $T_1 \sim T_2 (F)$, if for every $\varepsilon > 0$ there is a homeomorphism $h \in P_1$ onto $P_2$ such that the inequality $d[T_1 (p_1), T_2 h_1 (p_1)] < \varepsilon$ holds for each point $p_1 \in P_1$.

$T_1$ and $T_2$ are called Kerékjártó equivalent, written for brevity $T_1 \sim T_2 (K)$, if $T_1$ and $T_2$ admit of simultaneous monotone-light factorizations with the same middle-space and the same light factor.

We have (see Radó [1, p. 60]) that $T_1 \sim T_2 (F)$ implies that $T_1 \sim T_2 (K)$ whereas simple examples (see Radó [1, p. 60]) show that the converse is generally false. However, in certain special cases Kerékjártó equivalence does imply Fréchet equivalence. Two very important special cases used in surface area theory are the following.

(a) If $P_1$, $P_2$ are 2-cells and the common middle-space of the simultaneous monotone-light factorizations is a 2-cell, then (see Radó [1, p. 74]) $T_1 \sim T_2 (K)$ implies that $T_1 \sim T_2 (F)$.

(b) If $P_1$, $P_2$ are 2-spheres and the common middle-space of the simultaneous monotone-light factorizations is a 2-sphere, then (see Youngs [2]) $T_1 \sim T_2 (K)$ implies that $T_1 \sim T_2 (F)$.

If $P$ is a Peano space, then the uniform limit of a sequence of homeomorphisms from $P$ onto $P$ is a continuous monotone mapping from $P$ onto $P$. The results given in (a) and (b) above depend on the converse of this statement. That is to say, if $P$ is a 2-cell or a 2-sphere and $m$ is a continuous monotone mapping from $P$ onto $P$, then $m$ is the uniform limit of a sequence of homeomorphisms from $P$ onto $P$ (see Radó [1, pp. 71–72]). For extension of these results see Youngs [3]. In this note we give an example of a continuous monotone mapping of a 3-cell onto itself which is not the uniform limit of a sequence of homeomorphisms. This particular example enables us to show that in the 3-cell case Kerékjártó equivalence never implies Fréchet equivalence.

2. Preliminary lemmas. Let $P$ be a Peano space, let $I$ be the closed
unit interval \(0 \leq t \leq 1\), let \(I^0\) be the open interval \(0 < t < 1\), let \(T\) be a continuous mapping from \(P\) onto \(I\), and let (boldface arrows indicate a mapping onto) \(T = \lim, m:P \rightarrow \mathbb{R}, l: \mathbb{R} \rightarrow I\), be a monotone-light factorization of \(T\). For \(A = T^{-1}(0), B = T^{-1}(1), A^* = l^{-1}(0), B^* = l^{-1}(1)\) we set \(\alpha\) equal to the number of components \(O\) of \(P - (A + B)\) for which \(T(O) = I^0\) and \(\alpha^*\) equal to the number of components \(O^*\) of \(\mathbb{R} - (A^* + B^*)\) for which \(l(O^*) = I^0\).

**Lemma 1.** Under the above conditions \(0 < \alpha = \alpha^* < \infty\).

**Proof.** We have a simple arc \(\gamma\) with end points in \(A\) and in \(B\) and for which \(\gamma^0(A + B) = 0\), where \(\gamma^0\) is \(\gamma\) minus its end points. Then \(\gamma^0\) is in some component \(O\) of \(P - (A + B)\) and \(T(O) \supseteq T(\gamma^0) = I^0\). Therefore \(\alpha > 0\).

Assume that there is an infinite number \(O_1, O_2, \ldots\) of components of \(P - (A + B)\) such that \(T(O_n) = I^0\). Then the closures \(C_n\) of \(O_n\) are continua and \(T(C_n) = I\). We have a subsequence \(C_{n_j}\) which converges to a continuum \(C\) for which \(T(C) = I\). Let \(O\) be a component of \(P - (A + B)\). All but at most one of the \(C_{n_j}\) are in \(P - O\). Hence \(C \subseteq A + B\) and \(I = T(C) \subseteq T(A + B) = I - I^0\). This is a contradiction and hence \(0 < \alpha < \infty\).

By applying the above result to the mapping \(l\) we have \(0 < \alpha^* < \infty\). Since \(T = \lim\) and \(m\) is monotone, it follows that \(\alpha = \alpha^*\).

Now let \(P_1, P_2\) be homeomorphic Peano space, let \(T_1, T_2\) be continuous mappings from \(P_1, P_2\) respectively onto \(I\), and let \(\alpha_1\) and \(\alpha_2\) be defined as above.

**Lemma 2.** Under the above conditions if \(T_1 \sim T_2(F)\), then \(0 < \alpha_1 = \alpha_2 < \infty\).

**Proof.** Since \(T_1 \sim T_2(F)\), we have simultaneous monotone-light factorizations of the form \(T_1 = \lim, m_1: P_1 \rightarrow \mathbb{R}, l: \mathbb{R} \rightarrow I, T_2 = \lim, m_2: P_2 \rightarrow \mathbb{R}, l: \mathbb{R} \rightarrow I\). Let \(\alpha^*\) be defined as above for the mapping \(l\). By Lemma 1, \(\alpha_1 = \alpha^*\) and \(\alpha_2 = \alpha^*\). Therefore \(0 < \alpha_1 = \alpha_2 < \infty\).

3. A particular continuous monotone mapping. Let \(V: 0 \leq x^2 + y^2 + z^2 \leq 1\) be the solid unit sphere with center at the origin in \(xyz\)-space. Then \(V\) can be given in spherical coordinates

\[ V: x = r \cos \theta \sin \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \phi, \]

\[ 0 \leq r \leq 1, \ 0 \leq \theta \leq 2\pi, \ 0 \leq \phi \leq \pi, \]

where, for a point \((x, y, z)\), \(r\) is the distance of the point from the origin, \(\theta\) is the angle between the positive \(x\)-axis and the line from the origin to the point \((x, y, 0)\), and \(\phi\) is the angle between the positive
z-axis and the line from the origin to the point.

**Theorem 1.** There exists a continuous monotone mapping \( m : V \rightarrow V \) such that \( m \) is not the uniform limit of a sequence of homeomorphisms from \( V \) onto \( V \).

**Proof.** Using spherical coordinates let
\[
V' : x = r' \cos \theta' \sin \phi', \quad y = r' \sin \theta' \sin \phi', \quad z = r' \cos \phi',
\]
\[0 \leq r' \leq 1, \quad 0 \leq \theta' \leq 2\pi, \quad 0 \leq \phi' \leq \pi/2,
\]
be the solid hemisphere \( x'^2 + y'^2 + z'^2 \leq 1, \ z' \geq 0 \), in \( x'y'z' \)-space. We map \( V' \) continuously onto \( V \) by the relations \( r = r', \ \theta = \theta', \ \phi = 2\phi' \). Call this mapping \( M \). For \( (x, y, z) = (0, 0, 0) \neq (0, 0, -1 = z < 0) \), \( M^{-1}(x, y, z) \) is a single point. For \( (x, y, z) = (0, 0, -1 = z < 0) \), \( M^{-1}(0, 0, z) \) is the circle \( x'^2 + y'^2 = z^2 \) in the \( x'y' \)-plane. Thus \( M \) is a continuous monotone mapping from \( V' \) onto \( V \). Let \( h \) be any homeomorphism from \( V' \) onto \( V \). Then \( h(0, 0, 0) \) is a point on the surface of \( V \) and since \( M(0, 0, 0) \) is the point \( (0, 0, 0) \), it follows that the distance between \( h(0, 0, 0) \) and \( M(0, 0, 0) \) is one. Therefore \( M \) is not the uniform limit of a sequence of homeomorphisms from \( V' \) onto \( V \). If \( h^* \) is any homeomorphism from \( V \) onto \( V' \), then the same statement applies to the monotone mapping \( m = Mh^* \) from \( V \) onto \( V \).

4. **Main result.** For the solid sphere \( V \) given in \( \S 3 \) we shall denote by \( S \) the surface \( x^2 + y^2 + z^2 = 1 \) and for the solid hemisphere \( V' \) given in \( \S 3 \) we shall denote by \( H \) the surface \( x'^2 + y'^2 + z'^2 = 1, \ z' \geq 0 \), by \( K \) the surface \( 0 \leq x'^2 + y'^2 \leq 1, \ z' = 0 \) and by \( L \) the surface \( H + K \).

**Theorem 2.** Let \( T \) be a continuous mapping from \( V \) into a metric space \( \mathcal{P} \). If \( T(V) \) is not a single point, then there is a continuous mapping \( T^* \) from \( V \) into \( \mathcal{P} \) such that \( T \sim T^*(K) \) but such that \( T \) and \( T^* \) are not Fréchet equivalent.

**Proof.** Let \( T(0, 0, 0) = p_0 \) and \( T(0, 0, -1) = p_1 \). We may assume without loss of generality that \( p_0 \neq p_1 \). Let \( M \) be the continuous monotone mapping from \( V' \) onto \( V \) defined in Theorem 1. Set \( T' = TM \). Then \( T \sim T'(K) \). We assert that \( T \) and \( T' \) are not Fréchet equivalent. Assume that \( T \sim T'(F) \). Let \( T_S \) denote the mapping \( T \) operating from the surface \( S \), and let \( T^*_H, T^*_K, \) and \( T^*_L \) denote the mapping \( T' \) operating respectively from \( H, K, \) and \( L \). Since any homeomorphism from \( V' \) onto \( V \) takes \( L \) onto \( S \), the relation \( T \sim T'(F) \) implies that \( T_S \sim T^*_L(F) \).

Case 1. \( p_0 \notin T(S) \). Then \( T'(0, 0, 0) = T(0, 0, 0) = p_0 \notin T(S) = T_S(S) \). Thus \( T(S) \) and \( T'(L) \) are not the same point set and \( T_S \) and \( T^*_L \) are
not Fréchet equivalent. Therefore $T$ and $T'$ are not Fréchet equivalent.

Case 2. $p_0 \in T(S)$, $P$ is the unit interval $I : 0 \leq t \leq 1$ with $p_0 = 0$ and $p_1 = 1$. Let $\alpha_S$, $\alpha_L$, $\alpha_H$, and $\alpha_K$ be defined as in §3 respectively for the mappings $T_S$, $T'_L$, $T'_H$, and $T'_K$. If $M_H$ denotes the mapping $M$ operating from $H$, then $T'_H = T_S M_H$ and, since $M_H$ is monotone, it follows that $\alpha_S = \alpha_H$. Since $T'_H(x', y', 0) = 1$ for $x'^2 + y'^2 = 1$, it follows that $\alpha_L = \alpha_H + \alpha_K$. Since $\alpha_K > 0$, we have $\alpha_S < \alpha_L$. Therefore $\alpha_S \neq \alpha_L$, and, by Lemma 2, $T_S$ and $T'_L$ are not Fréchet equivalent. Therefore $T$ and $T'$ are not Fréchet equivalent.

Case 3. $p_0 \in T(S)$, $P$ an arbitrary metric space. For the metric $d(p', p'')$ let $t = f(p) = d(p, p_0)/[d(p, p_0) + d(p, p_1)]$. Then $f$ is a continuous mapping from $P$ onto $I$ and $T \sim T'(F)$ implies that $fT \sim fT'(F)$. Since $fT(0, 0, 0) = f(0, 0, -1) = f(p_1) = 1$, $fT$ and $fT'$ satisfy the conditions of Case 2. Thus $fT$ and $fT'$ are not Fréchet equivalent. Therefore $T$ and $T'$ are not Fréchet equivalent.

Now let $h$ be any homeomorphism from $V$ onto $V'$ and set $T^* = T'h$. Then $T' \sim T^*(F)$ and hence $T$ and $T^*$ are not Fréchet equivalent.

BIBLIOGRAPHY


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