A BOOLEAN ALGEBRA WITHOUT PROPER AUTOMORPHISMS

BJARNI JÓNSSON

It is the purpose of this note to show that there exists an infinite Boolean algebra which has no proper automorphisms.¹ We shall construct a simply ordered set $S$, introduce a topology on this set in the usual manner, the so-called interval topology determined by the ordering relation,² and prove that $S$ is a compact zero-dimensional Hausdorff space and that the only homeomorphism on $S$ onto $S$ is the identity mapping. It is well known³ that the group of automorphisms of the set-field $B$ consisting of all open and closed subsets of $S$ is isomorphic to the group of all homeomorphisms on $S$ onto $S$, whence it follows that $B$ has no proper automorphisms.

Consider a simply ordered set $S$ with at least two elements. By the interval topology on $S$ we mean the topology which has as a subbasis for open sets, the family of all sets $U\subseteq S$ such that either

$$U = \{ x \mid x \in S \text{ and } x < a \} \quad \text{or} \quad U = \{ x \mid x \in S \text{ and } a < x \}$$

for some $a \in S$. These sets together with all sets of the form

$$U = \{ x \mid x \in S \text{ and } a < x < b \}$$

with $a, b \in S$ constitute a basis for the interval topology on $S$. We shall need the following theorem.

**Theorem 1.** If $S$ is a simply ordered set, then the interval topology on $S$ is a Hausdorff topology. In order for $S$ to be compact and zero-dimensional, it is necessary and sufficient that the following conditions be satisfied:

(i) Every subset of $S$ has a least upper bound and a greatest lower bound in $S$.

(ii) Given any elements $a, b \in S$ with $a < b$, there exist elements $x, y \in S$ such that $a \leq x < y \leq b$, and such that $x \leq u \leq y$ implies that $u = x$ or $u = y$.

The proof of this theorem offers no difficulty and will be omitted. We shall use certain familiar concepts and results pertaining to

² Cf. G. Birkhoff, loc. cit. p. 60.
simply ordered sets. Suppose $S$ is a simply ordered set, and let $\omega_a$ and $\omega_\beta$ be regular initial ordinals (Anfangszahlen). An element $x \in S$ is said to have the character $(\omega_a, \omega_\beta)$ if there exist a strictly increasing transfinite sequence $y = (y_0, y_1, \ldots, y_t, \ldots)$ ($\xi < \omega_a$) such that $x$ is the least upper bound of the elements $y_t$ and a strictly decreasing transfinite sequence $z = (z_0, z_1, \ldots, z_n, \ldots)$ ($\eta < \omega_\beta$) such that $x$ is the greatest lower bound of the elements $z_n$. If $x$ is the least upper bound of a strictly increasing transfinite sequence $y = (y_0, y_1, \ldots, y_t, \ldots)$ and if the set of all elements $z \in S$ with $x < z$ is either empty or else has a smallest element, then $x$ is said to have the character $(\omega_a, 0)$. The phrase “$x$ has the character $(0, \omega_\beta)$” is defined in a similar manner. Finally, $x$ is said to have the character $(0, 0)$ if the set of all elements $y \in S$ with $y < x$ is either empty or else has a largest element and if the set of all elements $z \in S$ with $x < z$ is either empty or else has a smallest element.

It is well known that each element $x \in S$ has one and only one character. We now prove the following theorem.

**Theorem 2.** Suppose $S$ is a simply ordered set, $f$ maps $S$ homeomorphically onto itself (in the interval topology on $S$), and $x \in S$. We then have:

(i) If $x$ has the character $(\omega_a, \omega_\beta)$ where $\alpha \neq \beta$, then $f(x)$ has either the character $(\omega_a, \omega_\beta)$ or $(\omega_\beta, \omega_a)$.

(ii) If $x$ has one of the characters $(\omega_a, \omega_\beta)$, $(\omega_a, 0)$, $(0, \omega_\beta)$, then $f(x)$ also has one of these characters.

(iii) If $x$ has the character $(0, 0)$, then $f(x)$ also has the character $(0, 0)$.

**Proof.** Part (iii) is trivial since an element $x \in S$ has the character $(0, 0)$ if, and only if, $x$ is an isolated point in the interval topology of $S$.

Suppose $\omega_a$ is a regular initial ordinal and assume that there exists a strictly increasing transfinite sequence $y = (y_0, y_1, \ldots, y_t, \ldots)$ ($\xi < \omega_a$) such that $x$ is the least upper bound of the elements $y_t$. Then $x$ belongs to the closure of the set $A = \{ y_t \mid \xi < \omega_a \}$.

Furthermore, if $x$ belongs to the closure of the set $A' \subseteq A$, then $A'$ must have $\aleph_a$ elements. Let $B = \{ y \mid y \in A$ and $f(y) < f(x) \}$ and $C = \{ z \mid z \in A$ and $f(x) < f(z) \}$.

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4 Cf. F. Hausdorff, *Grundzüge der Mengenlehre*, Leipzig, 1914, chap. 6. Our definition of the character of an element differs slightly from the one given there. Thus we assign the characters $(\omega_a, 0)$, $(0, \omega_\beta)$, and $(0, 0)$ to elements which there are given the characters $(\omega_a, 1)$, $(1, \omega_\beta)$, and $(1, 1)$ respectively. We also find it convenient to assign characters to the end points of $S$ in case they exist.
Then $A = B \cup C$, so that $x$ belongs to the closure of either $B$ or $C$. Suppose $x$ belongs to the closure of $B$, then $f(x)$ belongs to the closure of $f(B)$, whence it follows that $x$ is the least upper bound of $f(B)$. Hence there exists a strictly increasing transfinite sequence $\gamma' = \langle y_0', y_1', \ldots, y_\xi', \ldots \rangle$ $(\xi < \omega_\alpha \leq \omega_\alpha)$ with $y_\xi' \in f(B)$ such that $f(x)$ is the least upper bound of the elements $y_\xi'$. It follows that $\alpha' = \alpha$.

Similarly, if $x$ belongs to the closure of $C$, then there exists a strictly decreasing transfinite sequence $\gamma = \langle z_0', z_1', \ldots, z_\eta', \ldots \rangle$ $(\eta < \omega_\alpha)$ with $z_\eta' \in f(C)$ such that $f(x)$ is the greatest lower bound of the elements $z_\eta'$.

From the above discussion we see that if $x$ has the character $(\omega_\alpha, \omega_\beta^*)$, then $f(x)$ has one of the characters $(\omega_\alpha, 0)$ or $(0, \omega_\alpha^*)$ or else $f(x)$ has a character of the form $(\omega_\alpha, \omega_\beta^*)$ or $(\omega_\beta, \omega_\alpha^*)$. A similar argument shows that $f(x)$ must have one of the characters $(\omega_\beta, 0)$ or $(0, \omega_\beta^*)$ or else $f(x)$ has a character of the form $(\omega_\beta, \omega_\alpha^*)$ or $(\omega_\alpha, \omega_\beta^*)$.

Since $f(x)$ has only one character, we infer that if $\alpha \neq \beta$, then $f(x)$ must have either the character $(\omega_\alpha, \omega_\beta^*)$ or $(\omega_\beta, \omega_\alpha^*)$. Thus (ii) holds.

Now suppose $x$ has one of the characters $(\omega_\alpha, \omega_\beta^*)$, $(\omega_\beta, 0)$, $(0, \omega_\beta^*)$. Then $f(x)$ has either one of these three characters or else $f(x)$ has a character of the form $(\omega_\alpha, \omega_\beta^*)$ or $(\omega_\beta, \omega_\alpha^*)$ with $\gamma_0' = \gamma$. However, applying (i) with $f$ and $x$ replaced by $f^{-1}$ and $f(x)$, we see that in the latter case $x$ would have one of the characters $(\omega_\alpha, \omega_\beta^*)$, $(\omega_\beta, \omega_\alpha^*)$.

Since this contradicts our assumption, the former case must apply, and (ii) holds. The proof is complete.

**Theorem 3.** There exists an infinite compact zero-dimensional topological space $S$ such that the only homeomorphism on $S$ onto $S$ is the identity mapping.

**Proof.** Let $\kappa_0 = 0$ and $\kappa_{n+1} = \omega_{\kappa_n}$ for $n = 0, 1, \ldots$, and for each $k < \omega_\alpha$ let $A_k$ be the family of all sequences $x = \langle x_0, x_1, \ldots, x_n, \ldots \rangle$ of ordinals $x_n$ such that $x_n < \kappa_{n+1}$ for $n = 0, 1, \ldots, k$ and $x_n = 0$ for $n = k+1, k+2, \ldots$. Then $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n \subseteq \cdots$, and the cardinal number of $A_k$ is $\aleph_k$. The set $B_k$ of all ordinals $\alpha$ with $\kappa_k \leq \alpha < \kappa_{k+1}$ also has $\aleph_k$ elements, hence there exists a univalent function $f_k$ on $A_k$ to $B_k$.

We next construct for each $k < \omega_\alpha$ a subset $C_k$ of $A_k$. Let $C_0 = A_0$, and assuming that $C_k$ has already been defined, consider a sequence $x \in A_{k+1}$. Then the sequence $x' = \langle x_0, x_1, \ldots, x_k, 0, 0, \ldots \rangle$ is a member of $A_k$. We let $x \in C_{k+1}$ if and only if $x' \in C_k$ and $x_{k+1} < \omega_\alpha$ where $\alpha = f_k(x') + 1$. Thus $C_k$ is defined for each $k < \omega_\alpha$; we denote the set-theoretical union of all the sets $C_k$ by $C$.

Let $\leq$ be the lexicographic ordering of $C$; that is, given two se-
sequences \( x, y \in C \), we let \( x \leq y \) if, and only if, \( x = y \) or else \( x_k < y_k \) where \( k \) is the smallest natural number such that \( x_k \neq y_k \). This relation simply orders \( C \) and well orders each of the sets \( C_k \).

Let \( D_0 \) be the set of all ordered pairs \( \langle x, 0 \rangle \) with \( x \in C \) such that \( x \neq \langle 0, 0, 0, \ldots \rangle \) and the last nonzero term of \( x \) is not a limiting ordinal, let \( D_1 \) be the set of all ordered pairs \( \langle x, 1 \rangle \) with \( x \in C \), and let \( D = D_0 \cup D_1 \). For two ordered pairs \( \langle x, i \rangle, \langle y, j \rangle \) in \( D \) we write \( \langle x, i \rangle \leq \langle y, j \rangle \) if, and only if, either \( x < y \) or else \( x = y \) and \( i \leq j \).

The simply ordered set \( D \) can be imbedded in a simply ordered set \( S \) such that every subset of \( S \) has a least upper bound and a greatest lower bound in \( S \), and such that each element \( p \in S \) is the least upper bound of all elements \( q \in D \) with \( q \leq p \) and the greatest lower bound of all elements \( r \in D \) with \( r \leq p \). We shall prove that the interval topology on \( S \) satisfies the conditions of the theorem.

The set \( C \) is dense in itself. In fact, suppose \( x, y \in C \) and \( x < y \). If \( n \) is the smallest natural number such that \( x^n = y^n \), then \( x_n < y_n \), the sequence \( z = \langle x_0, x_1, \ldots, x_n, x_{n+1} + 1, 0, 0, \ldots \rangle \) is a member of \( C \), and \( x < z < y \).

Suppose \( p, q \in S \) and \( p < q \). Then \( p \leq r < q \leq q \) for some \( r, q \in D \), and we have \( r = \langle x, i \rangle \) and \( q = \langle y, j \rangle \) with \( x, y \in C \) and \( i, j = 0, 1 \). If \( x = y \), then \( i = 0 \) and \( j = 1 \), and there exists no element \( t \in S \) with \( r < t < q \). If \( x < y \), then \( x < z < y \) for some \( z \in C \). First suppose the last nonzero term \( z_n \) of \( z \) is a limiting ordinal. Then \( z \) is the least upper bound of the sequences \( \langle z_0, z_1, \ldots, z_{n-1}, \alpha + 1, 0, 0, \ldots \rangle \) with \( \alpha < z_n \). Hence one of these sequences must be larger than \( x \). We therefore see that \( z \) can be so chosen that the last nonzero term is not a limiting ordinal. Letting \( r' = \langle z, 0 \rangle \) and \( s' = \langle z, 1 \rangle \), we thus have \( r', s' \in D \subseteq S \) and \( p \leq r' < s' \leq q \), and there exists no element \( t \in S \) with \( r' < t < s' \). We conclude by Theorem 1 that \( S \) is compact and zero-dimensional.

From the discussion in the preceding paragraph it is clear that the set \( D_0 \) is everywhere dense in \( S \) in the interval topology. In order to prove that the only homeomorphism on \( S \) onto \( S \) is the identity mapping, it is therefore sufficient to show that every homeomorphism on \( S \) onto \( S \) maps each member of \( D_0 \) onto itself. For this purpose we study the characters of the elements of \( S \).

Suppose \( p \in D_0 \). Then \( p = \langle x, 0 \rangle \) where \( x \in C \) and the last nonzero term \( x_n \) of \( x \) is not a limiting ordinal. Hence \( x_n = u + 1 \) for some ordinal \( u \). The sequence \( x' = \langle x_0, x_1, \ldots, x_{n-1}, u, 0, 0, \ldots \rangle \) is then a member of \( C \). Letting \( \alpha = f_n(x') + 1 \) and \( y^0 = \langle x_0, x_1, \ldots, x_{n-1}, u, \beta, 0, 0, 0, \ldots \rangle \) for every \( \beta < \omega_n \), we see that all the sequences \( y^0 \) are members of \( C \), and that their least upper bound is \( x \). Hence \( p \) is the least upper bound of all the ordered pairs \( \langle y^0, 1 \rangle \) with \( \beta < \omega_n \), and
we conclude that \( p \) has the character \((\omega_a, 0)\). Recalling how the functions \( f_a \) were chosen, we see that distinct members of \( D_0 \) have different characters.

Suppose \( p \in D_1 \). Then \( p = (x, 1) \) with \( x \in C \). Choosing \( n \) sufficiently large we have \( x_n = x_{n+1} = \cdots = 0 \). For each \( k < \omega_0 \) we let \( z^k = (z_0^k, z_1^k, \cdots, z_m^k, \cdots) \) where \( z_m^k = x_m \) for \( m < n \), \( z_{n+k}^k = 1 \), and \( z_m^k = 0 \) for \( n \leq m \neq n+k \). Then \( z^0 > z_1 > \cdots > z^k \cdot \cdots \), and the greatest lower bound of the sequences \( z^k \) is \( x \). Hence the greatest lower bound of the ordered pairs \((x^k, 1)\) is \( p \). We thus see that \( p \) has either the character \((0, \omega_0^*)\) or else a character of the form \((\omega_a, \omega_0^*)\).

Suppose \( p \in S - D \). First assume that \( p \) is the largest element of \( S \). Observe that the sequences \( y^k = (k, 0, 0, \cdots) \) with \( k < \omega_0 \) have no upper bound in \( C \). Therefore the least upper bound of the ordered pairs \((y^k, 1)\) is \( p \), whence it follows that the character of \( p \) is \((\omega_0, 0)\).

Next suppose \( p \) is not the largest element of \( S \). Then the set \( E \) of all elements \( q \in D_1 \) with \( p < q \) is nonempty. In fact, if we let \( E_k \) be the set of all ordered pairs \( q = (x, 1) \) with \( x \in C_k \), then \( E \cap E_k \) is nonempty and hence each of the sets \( E \cap E_k \) is nonempty. Since \( E_k \) is well ordered, each of the sets \( E \cap E_k \) has a smallest element \( q_k \). Hence \( q^0 \geq q^1 \geq \cdots \geq q^k \geq \cdots \), and the greatest lower bound of the elements \( q^k \) is \( p \). Hence \( p \) has a character of the form \((\omega_a, \omega_0^*)\).

Suppose \( \phi \) is a homeomorphism on \( S \) onto \( S \), and consider an element \( p \in D_0 \). Then \( p \) has a character of the form \((\omega_a, 0)\) with \( \alpha > 0 \), whence it follows by Theorem 2 that \( \phi(p) \) has one of the characters \((\omega_a, 0)\), \((0, \omega_0^*)\), \((\omega_a, \omega_0^*)\). But no other member of \( S \) has one of these characters, so that we must have \( \phi(p) = p \). Since this holds for every \( p \in D_0 \), we conclude that \( \phi \) is the identity mapping. The proof is complete.

We now obtain the following theorem.

**Theorem 4.** There exists an infinite Boolean algebra which has no proper automorphisms.

**Proof.** This is proved by Theorem 3 and the introductory remarks.

Brown University