ON THE INTEGRATION SCHEME OF MARÉCHAL

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J. E. Wilkins, Jr.\(^1\) proves the following assumption of A. Maréchal:\(^2\)

Let \(f(x, y)\) be a function of 2 variables, continuous in the interior of the circle \(C\), of radius \(R\); then

\[
\int \int_{C} f(x, y) \, d\sigma = \lim_{a \to 0} \left\{ 2\pi a \int_{S_a} f(x, y) \cdot ds \right\},
\]

where the double integral extends over the area of \(C\) and the line integral is taken along the arc of the archimedean spiral \((S_a)\)

\[
(S_a) \quad r = a\phi
\]

interior to \(C\).

In what follows, we give a short, elementary proof of (1), and two extensions.

I. Proof of (1). Let \(x = r\cos\phi, y = r\sin\phi\) and use the notations:

\[
A(m, n) = \frac{1}{2\pi} \int_{0}^{2\pi} \cos^{m} \phi \sin^{n} \phi \, d\phi,
\]

\[
B(m, n) = \frac{1}{\pi} \int_{0}^{\pi} \cos^{m} \phi \sin^{n} \phi \, d\phi,
\]

\[
C(m) = \int_{0}^{R} r \, dr.
\]

Then, in (1), \(d\sigma = r \, dr \, d\phi\) and \(ds = (r^2 + a^2)^{1/2} \cdot d\phi\).

As any continuous function can be approximated by a uniformly convergent sequence of polynomials, it is sufficient to prove (1) for \(f(x, y) = x^m y^n = r^{m+n} \cos^{m} \phi \sin^{n} \phi\). If \(m = n = 0\), (1) is verified by direct integration. If \(m + n > 0\), the first member of (1) becomes

\[
\int_{0}^{R} \int_{0}^{2\pi} r^{m+n+1} \cos^{m} \phi \sin^{n} \phi \, d\phi \, dr = 2\pi A(m, n) \cdot C(m + n + 1);
\]

and the second member may be written as

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\[ \lim_{a \to 0} \left\{ 2\pi a \int_{S_{a}} r^{m+n}(r^{2} + a^{2})^{1/2} \cos^{m} \phi \sin^{n} \phi d\phi \right\} \]

\[ = \lim_{a \to 0} 2\pi \int_{0}^{R} \left\{ r^{m+n+1} \cos^{m}(\frac{\nu}{a}) \cdot \sin^{n} \left( \frac{\nu}{a} \right) + O(a) \right\} dr. \]

In the (finite) Fourier expansion \( \cos^{m}(\frac{\nu}{a}) \cdot \sin^{n}(\frac{\nu}{a}) = a_{0}/2 + \sum_{r=1}^{n} (a_{r} \cos \left( \frac{\nu r}{a} \right) + b_{r} \sin \left( \frac{\nu r}{a} \right)) \) the first term is \( a_{0}/2 = (1/2\pi) \int_{0}^{\pi} \cos^{m} \phi \sin^{n} \phi d\phi = A(m, n). \) When \( a \to 0, \) then \( \nu \to \infty, \) so that, by the Riemann-Lebesgue lemma on Fourier series, \( \lim_{a \to 0} \int_{0}^{\pi} r^{m+n+1} \cos \left( \frac{\nu r}{a} \right) dr = 0, \) \( \lim_{a \to 0} \int_{0}^{\pi} r^{m+n+1} \sin \left( \frac{\nu r}{a} \right) dr = 0, \)

\( \nu = 1, 2, \ldots, m + n, \)

and (5) reduces to \( \lim_{a \to 0} \left\{ 2\pi \int_{0}^{R} r^{m+n+1} A(m, n) dr + 2\pi R \cdot O(a) \right\} = 2\pi A(m, n) \cdot C(m+n+1), \) the same as (4), proving (1).

II. Consider the integral of the continuous function \( f(x, y, z) \) taken on the surface of the sphere \( S \) of radius \( R. \) We may approximate it by the integral taken on a narrow strip, winding around the sphere, along the path

\[ (2') \]

\[ R \phi = a \theta, \]

from one pole (\( \phi = \theta = 0 \)) to the other (\( \phi = \pi, \theta = R \pi/a \)). We take the width of the strip to be \( 2\pi a \) and then make \( a \) tend to zero. The relation similar to (1) which we want to prove is, therefore,

\[ \int \int_{S} f(x, y, z) d\sigma = \lim_{a \to 0} \left\{ 2\pi a \int_{S_{a}} f(x, y, z) ds \right\}. \]

As before, it is sufficient to prove (6) for \( f(x, y, z) = x^{m}y^{n}z^{k}, m+n+k > 0, \) because, for \( m=n=k=0, f(x, y, z)=1 \) and (6) is verified by direct integration. The first member of (6) becomes successively, using (3), \( \int_{S} R^{m+n+k} \sin^{m+n} \phi \cos^{k} \phi \cos^{m} \theta \sin^{n} \theta \cdot R^{2} \sin \phi d\phi d\theta = R^{m+n+k+2} \int_{0}^{\pi} \cos^{m} \theta \sin^{n} \theta d\theta \int_{0}^{\pi} \sin^{m+n+1} \phi \cos^{k} \phi d\phi = R^{m+n+k+2} \cdot 2\pi A(m, n) \cdot \pi B(k, m+n+1) = 2\pi R^{m+n+k+1} A(m, n) \cdot B(k, m+n+1). \)

The second member of (6) becomes, as under I,

\[ \lim_{a \to 0} \left\{ 2\pi a \int_{S_{a}} R^{m+n+k} \sin^{m+n} \phi \cos^{k} \phi \cos^{m} \theta \sin^{n} \theta \cdot (R \sin \phi + O(a)) d\theta \right\}, \]

\[ = \lim_{a \to 0} \left\{ 2\pi a \cdot R^{m+n+k+1} \int_{S_{a}} \sin^{m+n+1} \phi \cos^{k} \phi \cos^{m} \theta \sin^{n} \theta d\theta + O(a) \right\}. \]
By (2'), \( \theta = \phi R/a \), so that the last expression becomes

\[
\lim_{a \to 0} \left\{ 2\pi R^{m+n+k+2} \int_{0}^{\pi} \sin^{m+n+1} \phi \cos^{k} \phi \cos^{m} (R\phi/a) \cdot \sin^{n} (R\phi/a) d\phi \right\}.
\]

Here \( g(\phi) = \sin^{m+n+1} \phi \cos^{k} \phi \) is a continuous, bounded function and we use, as under I, the relation

\[
\cos^{m} (R\phi/a) \cdot \sin^{n} (R\phi/a) = a_0/2 + \sum_{r=1}^{m+n} (a_r \cos (R\phi/a) + b_r \sin (R\phi/a))
\]

with \( a_0/2 = A(m, n) \). When \( a \to 0 \), \( R\phi/a \to \infty \) and it follows from the Riemann-Lebesgue lemma that all the expressions of the form

\[
\lim_{a \to 0} \int_{0}^{\pi} g(\phi) \cos (R\phi/a) d\phi \quad \text{and} \quad \lim_{a \to 0} \int_{0}^{\pi} g(\phi) \sin (R\phi/a) d\phi,
\]

\( \nu = 1, 2, \ldots, m + n \)

vanish and (8) reduces to

\[
2\pi R^{m+n+k+2} A(m, n) \int_{0}^{\pi} \sin^{m+n+1} \phi \cos^{k} \phi d\phi
\]

\[
= 2\pi^2 R^{m+n+k+3} A(m, n) \cdot B(k, m + n + 1),
\]

same as (7), proving (6).

III. Let the sphere \( S^r \), of radius \( r_r \), be covered by a wire of square section \( 2x \times 2x \), winding on the sphere along a spiral like \( (S') \). The outer surface of the wire is a new sphere of radius \( r_{r+1} = r_r + 2\pi a \)
and let this be covered in the same way, by the same wire, and so forth. In particular, making \( a \to 0 \), we can fill the interior of the sphere \( S \), of radius \( R \), with such successive layers of wire, winding along spirals of equations \( (2'') \)

\[
(S_a^r) \quad r_a = a \theta, \quad \nu = 1, 2, \ldots, \lceil R/2\pi a \rceil,
\]

where, in the \( \nu \)th layer from the center, \( r_\nu = \nu(2\pi a) \). We may attempt to approximate an integral, extended over the volume of the sphere, by the sum of integrals taken along the \( (S_a^r) \), which wind around the successive spherical shells, and are led to consider the equality

\[
\int \int \int_{r_\nu} f(x, y, z) dr = \lim_{a \to 0} \left\{ 4\pi^2 a^2 \sum_{r=1}^{\lceil R/2\pi a \rceil} \int_{S_a^r} f(x, y, z) \cdot ds \right\},
\]

where the integral in the first member is extended over the volume of

\[ [R/2\pi a] \text{ stands for the largest integer not exceeding } R/2\pi a. \]
ON THE DENSITY THEOREM

A. PAPOULIS

1. Introduction. Let $F$ be a set on the plane and $x$ a point of $F$. With $\{I_n\}$ an arbitrary sequence of intervals\(^1\) containing the point $x$ and with diameter tending to zero, we form the sequence $|F \cdot I_n|/|I_n|$.\(^2\) It has been shown (see [1] and [2])\(^3\) that for almost\(^4\) all points $x$ of $F$,

$$\lim_{I_n} \frac{|F \cdot I_n|}{|I_n|} = 1. \tag{1}$$

If the sequence $\{I_n\}$ of intervals is replaced by a sequence of arbitrary rectangles with sides not necessarily parallel to the axes of coordinates, then the above ceases to be true. H. Busemann and W. Feller (see [1]) have shown that if the direction of some one of the sides of the rectangles $\{I_n\}$ varies within any nonzero angle, then (1) is no longer true for all sets $F$.

The purpose of the following is to show that even if the direction of the rectangles $\{I_n\}$ converging to the point $x$ is fixed, then (1) is still not true for some sets, provided of course that the fixed direction may vary from point to point.

\(^1\) Rectangles with sides parallel to the coordinate axes.
\(^2\) The number $|E|$ will mean the two-dimensional Lebesgue-measure of the set $E$.
\(^3\) Numbers in brackets refer to the references at the end of the paper.
\(^4\) By “almost all points $x$ of a set $E$” we shall mean all points of $E$ except for a set of measure zero; this will also be indicated by p.p.