POLYNOMIAL ITERATIONS TO ROOTS
OF ALGEBRAIC EQUATIONS
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If $\xi$ is a root of the equation
(1) $f(x) = 0,$
and if $\phi(x)$ satisfies
(2) $\phi(\xi) = \xi, \quad \phi^{(s)}(\xi) = 0 \quad (s = 1, 2, \ldots, r - 1),$
then $\phi(x)$ is said to define an iteration of order $r$ to the root $\xi$. In
fact, for $r > 1$, when $x_0$ is in a sufficiently small neighborhood of $\xi$ the
sequence
(3) $x_{i+1} = \phi(x_i)$
converges to $\xi$ with
(4) $x_{i+1} = \xi + O(x_i - \xi)^r$.
For analytic $f$, iterations of all orders exist and can be constructed in
many ways. Domb [2]1 has shown further that for polynomial $f$ it is
always possible to make $\phi$ a polynomial. The purpose of this note is
to describe a simple algorithm:
Let $f(x)$ be a polynomial with no multiple factors; let $p(x)$ and $q(x)$
be any polynomials satisfying
(5) $pf' + qf = 1;
let
(6) $\phi_1 = x - pf, \quad \phi_r = \phi_{r-1} + p_r(-f)^r/r!,$
$\rho_1 = p, \quad \rho_r = \rho_{r-1} - (r - 1)q\rho_{r-1}.$
Then $\phi_r(x)$ is clearly a polynomial, and it defines an iteration of order
$r + 1$ to any root $\xi$ of the equation (1).
While a direct induction is possible, it is simpler to proceed other-
wise. Schröder [3] (see also [1]) has shown that
(7) $\phi_r = x + \sum_{1}^{r} (-F)^r F^{-1} F'/r!,$
$\delta^r F = 1/F', \quad \delta^{r+1} F = (\delta^r F)'/F'.

1 Numbers in brackets refer to the bibliography at the end of the paper.

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defines an iteration of order \( r+1 \) to the roots of
\[ F = 0. \]

We ask, then, whether a function \( g(x) \) can be found such that for
\[ (8) \quad F = gf \]
the functions \( \phi_r \) defined by (7) are polynomials. For \( r=1 \) we require that
\[ (9) \quad F/F' = pf \]
for some polynomial \( p \). To achieve this it is sufficient to choose \( p \) and \( f \) to satisfy (5) while
\[ (10) \quad g'/g = q/p. \]
Suppose, now, it has been verified that
\[ F^{r+1}F = p_r f^r, \]
or, by (8), that
\[ \delta^{r+1}F = p_r g^{r-1}. \]
Then
\[ \delta^rF = (p'_r g^{r-1} - r p_r g^{r-2})/F', \]
and after a few manipulations
\[ F^{r+1}F = f^{r+1}(p p'_r - r q p_r). \]
Hence
\[ p_{r+1} = p p'_r - r q p_r \]
and the induction is complete.

\[ \text{BIBLIOGRAPHY} \]


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