POLYNOMIAL ITERATIONS TO ROOTS OF ALGEBRAIC EQUATIONS

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If ξ is a root of the equation

$$f(x) = 0, \tag{1}$$

and if $\phi(x)$ satisfies

$$\phi(\xi) = \xi, \qquad \phi^{(s)}(\xi) = 0 \qquad (s = 1, 2, \cdots, r-1), \tag{2}$$

then $\phi(x)$ is said to define an iteration of order r to the root ξ. In fact, for $r > 1$, when x_0 is in a sufficiently small neighborhood of ξ the sequence

$$x_{i+1} = \phi(x_i) \tag{3}$$

converges to ξ with

$$x_{i+1} = \xi + O(x_i - \xi)^r. \tag{4}$$

For analytic f, iterations of all orders exist and can be constructed in many ways. Domb [2][1] has shown further that for polynomial f it is always possible to make ϕ a polynomial. The purpose of this note is to describe a simple algorithm:

Let $f(x)$ be a polynomial with no multiple factors; let $p(x)$ and $q(x)$ be any polynomials satisfying

$$pf' + qf \equiv 1; \tag{5}$$

let

$$\begin{aligned} \phi_1 &= x - p_1 f, & \phi_r &= \phi_{r-1} + p_r(-f)^r/r!, \\ p_1 &= p, & p_r &= pp'_{r-1} - (r-1)qp_{r-1}. \end{aligned} \tag{6}$$

Then $\phi_r(x)$ is clearly a polynomial, and it defines an iteration of order $r+1$ to any root ξ of the equation (1).

While a direct induction is possible, it is simpler to proceed otherwise. Schröder [3] (see also [1]) has shown that

$$\begin{aligned} \phi_r &= x + \sum_1^r (-F)^\nu \delta^{\nu-1} F/\nu!, \\ \delta^0 F &= 1/F', \qquad \delta^{\nu+1} F = (\delta^\nu F)'/F', \end{aligned} \tag{7}$$

Presented to the Society, November 25, 1950; received by the editors October 23, 1950.

[1] Numbers in brackets refer to the bibliography at the end of the paper.

defines an iteration of order $r+1$ to the roots of

$$F = 0.$$

We ask, then, whether a function $g(x)$ can be found such that for

$$F = gf \tag{8}$$

the functions ϕ_r defined by (7) are polynomials. For $r=1$ we require that

$$F/F' = pf \tag{9}$$

for some polynomial p. To achieve this it is sufficient to choose p and f to satisfy (5) while

$$g'/g = q/p. \tag{10}$$

Suppose, now, it has been verified that

$$F^r\delta^{r-1}F = p_r f^r ,$$

or, by (8), that

$$\delta^{r-1}F = p_r g^{-r}.$$

Then

$$\delta^r F = (p_r' g^{-r} - r p_r g' g^{-r-1})/F',$$

and after a few manipulations

$$F^{r+1}\delta^r F = f^{r+1}(p p_r' - r q p_r).$$

Hence

$$p_{r+1} = p p_r' - r q p_r$$

and the induction is complete.

Bibliography

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2. C. Domb, *On iterative solutions of algebraic equations*, Proc. Cambridge Philos. Soc. vol. 45 (1949) pp. 237–240.

3. Ernst Schröder, *Über unendliche viele Algorithmen zur Auflösung der Gleichungen*, Math. Ann. vol. 2 (1870) pp. 317–365.

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