FUNCTIONS WHICH REPRESENT PRIME NUMBERS

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W. H. Mills [1] has proved that there is a real number \( A \) so that \([A^n]\) is a prime for every positive integer \( n \). L. Kuipers [2] has extended this and proved that for any positive integer \( c \geq 3 \) there is a real number \( A = A(c) \) so that

\[
[A^n] \text{ is a prime of every positive integer } n.
\]

We generalize this to the following result.

**Theorem.** Given any real number \( c > 8/3 \), there exists a corresponding real number \( A \) with property (1). Furthermore, given any real number \( A > 1 \), there exists a value \( c \) with property (1).

The proof of the first part of the theorem is a slight rearrangement of Mills' proof, and the proof of the second part employs the same basic idea in a different setting.

We begin by noting that if \( x > 1 \) and \( y \geq 2 \), then

\[
(1 + x)^y > x^y + x^{y-1} + 1.
\]

Also we use, as did Mills and Kuipers, the well known result of Ingham [3] that there is a constant \( K \) such that for \( x > 1 \) there is a prime \( p \) satisfying

\[
x < p < x + Kx^{6/5}.
\]

**Proof of Part 1.** Write \( c = 8(1 + d)/3 \), and choose any prime \( p_1 > K^{1/d} \). For \( n = 1, 2, 3, \ldots \) choose the primes \( p_{n+1} \) to satisfy

\[
p_n < p_{n+1} < p_n + K p_{n}^{6/5}.
\]

Now \( K p_n^{6/5} < p_n^{6/5} < p_n^{6/5} + K p_n^{6/5} = p_n^{6/5} \), and using (2) we can rewrite (4) as

\[
p_n^c < p_{n+1} < p_n^c + p_n^{c-1} < (1 + p_n)^c - 1.
\]

Now this in turn implies

\[
p_n^{c^n} < p_{n+1}^{c^{n+1}} < (1 + p_{n+1})^{c^{n+1}} - 1 < (1 + p_n)^{c^n},
\]

so that \( p_n^{c^n} \) is a bounded monotonic increasing sequence, whose limit

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1 Numbers in brackets refer to the references at the end of the paper.

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we denote by $A$. Thus (6) implies

$$p_n < A < (1 + p_n)^{e_n} \text{ or } p_n < A^{e_n} < 1 + p_n.$$  

**Proof of Part 2.** Starting again, we now choose the prime $p_1$ to satisfy $p_1 > A^k$ and $p_1 > K$, and we shall prove that we can find an infinite sequence of primes $p_2, p_3, \ldots$ to satisfy the recursive inequality

$$A^{\log p_n} < A^{\log (1 + p_n)} < A^{\log (1 + p_n)}(n+1)^{1/n} - 1,$$

all logarithms being to base $A$. This can be written in the form

$$p_n^{(\log (1 + p_n))^{1/n}} < p_{n+1} < (1 + p_n)^{(\log (1 + p_n))^{1/n}} - 1.$$  

We use induction to prove that these primes can be so chosen. Thus we shall assume that $p_2, p_3, \ldots, p_n$ have been obtained, and noting that the first part of inequality (8) can be written as

$$(\log p_n)^{1/n} < (\log p_{n+1})^{1/(n+1)},$$

we have the inequality $\log p_n > (\log p_1)^n$ from our induction hypothesis. Hence the following inequalities can be written:

$$\{\log (1 + p_n)\}^{1/n} > \{\log p_n\}^{1/n} > \log p_1 > 8,$$

$$p_n^{(\log (1 + p_n))^{1/n}/8} > p_n > p_1 > K.$$  

Using (2), (10), and (11) we have

$$(1 + p_n)^{(\log (1 + p_n))^{1/n}} - 1 > p_n^{(\log (1 + p_n))^{1/n}/8} + p_n^{(\log (1 + p_n))^{1/n}/8} + p_n^{(\log (1 + p_n))^{1/n}/8} + Kp_n^{(\log (1 + p_n))^{1/n}/8}.$$  

This establishes, in the light of (3), the existence of the sequence of primes satisfying (9) or (8), and we next observe that (8) implies

$$\{\log p_n\}^{1/n} < \{\log p_{n+1}\}^{1/(n+1)} < \{\log (1 + p_n)\}^{1/(n+1)} < \{\log (1 + p_n)\}^{1/n}.$$  

Thus the bounded monotonic increasing sequence $\{\log p_n\}^{1/n}$ tends to a limit which we denote by $c$, so that

$$\{\log p_n\}^{1/n} < c < \{\log (1 + p_n)\}^{1/n} \text{ or } p_n < A^{e_n} < 1 + p_n.$$  

Thus the theorem is proved, and it can be observed that there are infinitely many values of $A$ for each value of $c$, and vice versa, because of the freedom in the choice of $p_1$.  

References


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