FUNCTIONS WHICH REPRESENT PRIME NUMBERS

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W. H. Mills [1][1] has proved that there is a real number A so that $[A^{3^n}]$ is a prime for every positive integer n. L. Kuipers [2] has extended this and proved that for any positive integer $c \geqq 3$ there is a real number $A = A(c)$ so that

$$[A^{c^n}] \text{ is a prime of every positive integer } n. \tag{1}$$

We generalize this to the following result.

THEOREM. *Given any real number $c > 8/3$, there exists a corresponding real number A with property* (1). *Furthermore, given any real number $A > 1$, there exists a value c with property* (1).

The proof of the first part of the theorem is a slight rearrangement of Mills' proof, and the proof of the second part employs the same basic idea in a different setting.

We begin by noting that if $x > 1$ and $y \geqq 2$, then

$$(1 + x)^y > x^y + x^{y-1} + 1. \tag{2}$$

Also we use, as did Mills and Kuipers, the well known result of Ingham [3] that there is a constant K such that for $x \geqq 1$ there is a prime p satisfying

$$x < p < x + Kx^{5/8}. \tag{3}$$

PROOF OF PART 1. Write $c = 8(1+d)/3$, and choose any prime $p_1 > K^{1/d}$. For $n = 1, 2, 3, \cdots$ choose the primes p_{n+1} to satisfy

$$p_n^c < p_{n+1} < p_n^c + Kp_n^{5c/8}. \tag{4}$$

Now $Kp_n^{5c/8} < p_1^d \cdot p_n^{5c/8} < p_n^{d+5c/8} = p_n^{c-1}$, and using (2) we can rewrite (4) as

$$p_n^c < p_{n+1} < p_n^c + p_n^{c-1} < (1 + p_n)^c - 1. \tag{5}$$

Now this in turn implies

$$p_n^{c^{-n}} < p_{n+1}^{c^{-n-1}} < (1 + p_{n+1})^{c^{-n-1}} < (1 + p_n)^{c^{-n}}, \tag{6}$$

so that $p_n^{c^{-n}}$ is a bounded monotonic increasing sequence, whose limit

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[1] Numbers in brackets refer to the references at the end of the paper.

we denote by A. Thus (6) implies

$$p_n^{c^{-n}} < A < (1+p_n)^{c^{-n}} \quad \text{or} \quad p_n < A^{c^n} < 1 + p_n.$$

PROOF OF PART 2. Starting again, we now choose the prime p_1 to satisfy $p_1 > A^8$ and $p_1 > K$, and we shall prove that we can find an infinite sequence of primes $p_2, p_3, \cdots$ to satisfy the recursive inequality

$$A^{\{\log p_n\}^{(n+1)/n}} < A^{\log p_{n+1}} < A^{\{\log(1+p_n)\}^{(n+1)/n}} - 1, \tag{8}$$

all logarithms being to base A. This can be written in the form

$$p_n^{\{\log p_n\}^{1/n}} < p_{n+1} < (1+p_n)^{\{\log(1+p_n)\}^{1/n}} - 1. \tag{9}$$

We use induction to prove that these primes can be so chosen. Thus we shall assume that $p_2, p_3, \cdots, p_n$ have been obtained, and noting that the first part of inequality (8) can be written as

$$(\log p_n)^{1/n} < (\log p_{n+1})^{1/(n+1)},$$

we have the inequality $\log p_n > (\log p_1)^n$ from our induction hypothesis. Hence the following inequalities can be written:

$$\{\log(1+p_n)\}^{1/n} > \{\log p_n\}^{1/n} > \log p_1 > 8, \tag{10}$$

$$p_n^{\{\log(1+p_n)\}^{1/n}/8} > p_n > p_1 > K. \tag{11}$$

Using (2), (10), and (11) we have

$$\begin{aligned}(1+p_n)^{\{\log(1+p_n)\}^{1/n}} - 1 &> p_n^{\{\log(1+p_n)\}^{1/n}} + p_n^{\{\log(1+p_n)\}^{1/n}-1} \\ &> p_n^{\{\log p_n\}^{1/n}} + p_n^{7\{\log(1+p_n)\}^{1/n}/8} \\ &> p_n^{\{\log p_n\}^{1/n}} + Kp_n^{5\{\log p_n\}^{1/n}/8}.\end{aligned}$$

This establishes, in the light of (3), the existence of the sequence of primes satisfying (9) or (8), and we next observe that (8) implies

$$\begin{aligned}\{\log p_n\}^{1/n} < \{\log p_{n+1}\}^{1/(n+1)} &< \{\log(1+p_{n+1})\}^{1/(n+1)} \\ &< \{\log(1+p_n)\}^{1/n}.\end{aligned}$$

Thus the bounded monotonic increasing sequence $\{\log p_n\}^{1/n}$ tends to a limit which we denote by c, so that

$$\{\log p_n\}^{1/n} < c < \{\log(1+p_n)\}^{1/n} \quad \text{or} \quad p_n < A^{c^n} < 1 + p_n.$$

Thus the theorem is proved, and it can be observed that there are infinitely many values of A for each value of c, and vice versa, because of the freedom in the choice of p_1.

References

1. W. H. Mills, *A prime-representing function*, Bull. Amer. Math. Soc. vol. 53 (1947) p. 604.

2. L. Kuipers, *Prime-representing functions*, Neder. Akad. Wetensch. vol. 53 (1950) pp. 309–310.

3. A. E. Ingham, *On the difference between consecutive primes*, Quart. J. Math. Oxford Ser. vol. 8 (1937) pp. 255–266.

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