Further \( \mathfrak{S} \) can, apart from trivial (unit) factors, be expressed in at most one way as a product of indecomposable factors. The same results hold for families of regular bilinear mappings \((n = 2, H = K, H_i = K_i)\) and for families of groups of class 1 or 2. If either \( Q \) or \( G/Z \) be finite, \( F(G) \) is uniquely expressible as a product of indecomposable families.

Reference

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**A SHORT PROOF OF AN IDENTITY OF EULER**

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Euler discovered the identity

\[
\prod_{s=1}^{n} (1 - x^s) = 1 + \sum_{s=1}^{\infty} (-1)^s [x^{s(2s-1)/2} + x^{s(2s+1)/2}].
\]

He used it in the theory of partitions, and, after some time, he proved it \([1]\). Later, famous proofs involving theta functions and combinatorial arguments were given by Jacobi and F. Franklin \([2]\). The following algebraic proof is quite simple.

Let the partial products and partial sums of (1) be

\[
P_0 = 1, \quad P_n = \prod_{s=1}^{n} (1 - x^s),
\]

and

\[
S_n = 1 + \sum_{s=1}^{n} (-1)^s [x^{s(2s-1)/2} + x^{s(2s+1)/2}].
\]

Then \( S_n \) and \( P_n \) are related by the *finite* identity

\[
S_n = F_n \quad \text{where} \quad F_n = \sum_{s=0}^{n} (-1)^s \frac{P_n}{P_s} x^{s+n(s+1)/2}.
\]

To prove (2) we detach the last term, \( s = n \), and split the remaining sum into two parts by putting \( P_n = P_{n-1} - x^n P_{n-1} \). This gives

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1 Numbers in brackets refer to the references cited at the end of the paper.
\[ F_n = \sum_{s=0}^{n-1} (-1)^s \frac{P_{n-1}}{P_s} x^{s(n+1)/2} + \sum_{r=0}^{n-1} (-1)^{s+1} \frac{P_{n-1}}{P_r} x^{r(n+r+1)/2} \]

We detach the term \( r = n - 1 \), and recombine the two sums by putting \( r = s - 1 \) and adding the new corresponding terms. By using \( 1/P_{s-1} = (1 - x^s)/P_s \) we obtain

\[ F_n = \sum_{s=0}^{n-1} (-1)^s \frac{P_{n-1}}{P_s} x^{s(n+1)/2} + (-1)^s \left[ x^{n(n-1)/2} + x^{n(n+1)/2} \right] \]

or, referring to the definitions of \( F_n \) and \( S_n \),

\[ F_n = F_{n-1} + (S_n - S_{n-1}) \]

Therefore \( S_n = F_n \) if \( S_{n-1} = F_{n-1} \) and since

\[ S_1 = 1 - (x + x^2) = (1 - x) - x^2 = F_1, \]

equation (2) follows by induction.

Now the partial product, \( P_n \), is the first term of \( F_n \) (\( s=0 \) in (2)), and since the remaining terms of \( F_n \) are of order \( x^{n+1} \) and higher, \( P_n \) must agree with \( F_n \) up to \( x^n \). Therefore \( P_n \) agrees with \( S_n \) up to \( x^n \) and, letting \( n \to \infty \), we see that both sides of (1) have the same power series. This proves the theorem.

The origin of the formula, \( F_n \), is of interest. It was obtained by applying a nonlinear transformation \([3]\) to the sequence of products, \( A_i = 1/P_i \). Given the sequence \( A_i \) (\( i = 0, 1, \ldots, 2k \)), one obtains the transform, \( B_{kk} \), from the formula:

\[ B_{kk} = \begin{vmatrix}
A_0 & A_1 & A_2 & \cdots & A_k \\
\Delta A_0 & \Delta A_1 & \cdots & \cdots & \Delta A_k \\
\Delta A_1 & \Delta A_2 & \cdots & \cdots & \Delta A_{k+1} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\Delta A_{k-1} & \cdots & \cdots & \cdots & \Delta A_{2k-1}
\end{vmatrix}
\]

where \( \Delta A_i = A_{i+1} - A_i \).

Reduction of the determinants gives \( B_{kk} = 1/F_k \). The members of the original sequence, \( A_0 \) to \( A_{2k} \), agree with the infinite series to, at most,
and since the transform $B_{kk}$ agrees to $x^{k(3k+5)/2}$, we see that the transformation greatly increases the rate of convergence of the sequence.

By a very similar calculation (proof omitted), one finds that if

$$Q_0 = 1, \quad Q_n = \prod_{i=1}^{n} \frac{(1 - x^{2i})}{(1 - x^{2i-1})}, \quad \text{and} \quad T_n = 1 + \sum_{i=1}^{n} [x^{2(2i-1)} + x^{2(2i+1)}],$$

then $T_n = G_n$ where $G_n = \sum_{i=0}^{n} (Q_n/Q_0) x^{2n+1}$. This implies $Q_\infty = T_\infty$, an identity due to Gauss [2].

**References**


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