FAMILIES OF LORENTZIAN MATRICES

C. C. MACDUFFEE

1. The differential equation. Let \( P(s) \) be an \( n \)-rowed square matrix whose elements are continuous real functions of the real variable \( s \) in an interval \( s_0 \leq s \leq s_1 \). The differential equation

\[
\Lambda'(s) = P(s) \cdot \Lambda(s), \quad \Lambda(s) = (\lambda_{ij}),
\]

is equivalent to the \( n \) systems of equations

\[
\lambda'_{1i} = p_{11}\lambda_{1i} + p_{12}\lambda_{2i} + \cdots + p_{1n}\lambda_{ni}, \\
\lambda'_{2i} = p_{21}\lambda_{1i} + p_{22}\lambda_{2i} + \cdots + p_{2n}\lambda_{ni}, \\
\vdots \\
\lambda'_{ni} = p_{n1}\lambda_{1i} + p_{n2}\lambda_{2i} + \cdots + p_{nn}\lambda_{ni}, \quad i = 1, 2, \ldots, n.
\]

The coefficients are independent of \( i \) so that each column of \( \Lambda(s) \) is a solution of this system.

These equations are known \([1; 2; 3]\) to have a solution for every choice of initial conditions, and to have exactly \( n \) linearly independent solutions, every solution then being a linear combination of any set of \( n \) linearly independent ones. This is equivalent to the statement in matrix notation that there is a solution \( \Lambda(s) \) such that \( |\Lambda(s_0)| \neq 0 \), and that every solution is then given by \( \Lambda(s) \cdot A \) where \( A \) is an arbitrary constant real matrix. If the elements of \( P(s) \) are \( n-1 \) times differentiable, it can be shown that each element of \( \Lambda(s) \) satisfies a linear differential equation whose coefficients are polynomials in the elements of \( P(s) \) and their derivatives.

If (1) has a solution \( \Lambda(s) \) all of whose elements are analytic functions of \( s \) in a neighborhood of \( s = 0 \), it may be expressed in power series. From (1) we have

\[
\Lambda'(0) = P(0) \cdot \Lambda(0), \\
\Lambda''(s) = P'(s) \cdot \Lambda(s) + P(s) \cdot \Lambda'(s) = [P'(s) + P^2(s)] \Lambda(s).
\]

Similarly

\[
\Lambda^{(i)}(s) = M_i(s) \cdot \Lambda(s)
\]

where \( M_i(s) \) is a polynomial in \( P(s) \) and its derivatives. Then near

Presented to the International Congress of Mathematicians, September 2, 1950 under the title Curves in Minkowski space; received by the editors November 1, 1950.

Numbers in brackets refer to the references at the end of the paper.

794
s = 0[4; 5]

\[ \Lambda(s) = \left[ I + M_1(0)s + \frac{1}{2} M_2(0)s^2 + \frac{1}{3!} M_3(0)s^3 + \cdots \right] \Lambda(0). \]

In the case where \( P \) is a constant matrix, this becomes particularly simple, for \( \Lambda^{(0)}(s) = P^t \Lambda(s) \) so that

\[ \Lambda(s) = \left[ I + P \cdot s + \frac{1}{2} P^2 \cdot s^2 + \frac{1}{3!} P^3 \cdot s^3 + \cdots \right] \Lambda(0) = e^{P \cdot s} \Lambda(0), \]

which is a known result. In fact, Hurewicz [3, p. 43] has shown that \( \Lambda(s) = e^{P \cdot s} \cdot \Lambda (0) \) whenever \( P(s) \) is commutative with its integral.

All the results of this section may be paralleled for the equation

\[ A'(s) = A(s) - Q(s) \]

by taking the transposed matrices.

2. Lorentzian matrices. We shall denote by \( J \) an \( n \) by \( n \) matrix whose elements are real numbers and which is both symmetric and orthogonal. Since \( J \) is symmetric, there exists a real orthogonal matrix \( O \) such that

\[ O^t J O = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = D \]

where \( \lambda_1, \lambda_2, \cdots, \lambda_n \) are the characteristic roots of \( J \), and are all real. Since \( J \) is orthogonal, these roots are of absolute value unity so that \( \lambda_i = \pm 1 \). Clearly

\[ J = J^t = J^{-1}, \quad |J| = \pm 1. \]

The set of all matrices \( A \) such that

\[ A^t J A = J \]

consitute a group which we shall call a Lorentzian group and denote by \( \mathcal{L}_J \). It is merely a matter of an isomorphism

\[ A \leftrightarrow O^t A O \]

to suppose that \( J \) is actually the diagonal matrix \( D \). If \( D = I \), then \( \mathcal{L}_I \) is the orthogonal group of \( n \) by \( n \) matrices so that the theory of orthogonal matrices is contained in this treatment.
Theorem 1. If $A \in \mathbb{S}$, then also $A^T \in \mathbb{S}$.

For if $A^TA = J$, then $A^TAJ = J^2 = I$ so that

$$AJ = (A^TA)^{-1} = J^{-1}A^{-T} = JA^{-T}.$$ 

That is, $AJAT = J$.

Theorem 2. Let $A(s)$ be a matrix whose elements are differentiate real functions of $s$ which is in $\mathbb{S}$ for every $s$ in the interval $s_0 \leq s \leq s_1$. Then there exist unique matrices $P(s)$ and $Q(s)$ such that $J \cdot P(s)$ and $Q(s) \cdot J$ are skew, and

$$
\Lambda'(s) = P(s) \cdot \Lambda(s) = \Lambda(s) \cdot Q(s).
$$

Upon differentiating $\Lambda(s) \cdot J \cdot \Lambda(s) = J$, we have

$$
\Lambda'(s) \cdot J \cdot \Lambda(s) + \Lambda(s) \cdot J \cdot \Lambda'(s) = 0.
$$

That is,

$$
\Lambda'(s) \cdot J \cdot \Lambda(s) + \Lambda(s) \cdot J \cdot \Lambda'(s) = 0,
$$

$$
\Lambda'(s) \cdot J \cdot \Lambda(s) + \Lambda(s) \cdot J \cdot \Lambda'(s) = 0.
$$

Since $|\Lambda(s)| = \pm 1$, $\Lambda(s)$ is nonsingular and

$$
J \cdot P(s) = - P'(s) \cdot J = -(J \cdot P(s))^T
$$

so that $J \cdot P(s)$ is skew.

Upon differentiating $\Lambda(s) \cdot J \cdot \Lambda'(s) = J$, we obtain similarly

$$
\Lambda'(s) = \Lambda(s) \cdot Q(s)
$$

where $Q(s) \cdot J$ is skew.

Clearly $P(s) \cdot J$ is skew if and only if $J \cdot P(s)$ is skew, and similarly $Q(s) \cdot J$ is skew if and only if $J \cdot Q(s)$ is skew.

Since for every $s$ in the interval $s_0 \leq s \leq s_1$, $|\Lambda(s)| = \pm 1$, it follows that $\Lambda'(s) \cdot \Lambda^{-1}(s) = P(s)$ so that $P(s)$ is unique. Similarly $Q(s) = \Lambda^{-1}(s) \cdot \Lambda'(s)$ is unique.

Theorem 3. Let $P(s)$ be a matrix whose elements are continuous real functions of $s$ in the interval $s_0 \leq s \leq s_1$ having the property that $P(s) \cdot J$ is skew. Let $\Lambda(s)$ be a matrix which satisfies a differential equation

$$
\Lambda'(s) = P(s) \cdot \Lambda(s),
$$

subject to the initial condition that $\Lambda(s_2) \in \mathbb{S}$ for some value $s_2$ of $s$ in the interval $s_0 \leq s \leq s_1$. Then $\Lambda(s) \in \mathbb{S}$ throughout the interval.

Set $\Lambda'(s) \cdot J \cdot \Lambda(s) = R(s)$. Then as before
\[ \Lambda^T(s) \left[ P^T(s) \cdot J + J \cdot P(s) \right] \Lambda(s) = R'(s). \]

Since \( J \cdot P(s) \) is skew, \( R'(s) = 0 \) and \( R(s) \) is a constant matrix. That is, \( R(s) \) has for every value of \( s \) in the interval the value that it has for \( s = s_2 \), namely \( J \). Thus \( \Lambda(s) \in \mathbb{R}_f \).

**References**