SOME THEOREMS ON MEROMORPHIC FUNCTIONS

S. M. SHAH

1. **Introduction.** In a recent paper [1][1] Yoshitomo Okada proved the following two theorems.

THEOREM A. *If for any meromorphic function*

$$F(z) = f(z)/g(z), \tag{1}$$

where f and g are canonical products of genera p, q *and of orders* ρ_1, ρ_2 *respectively,*

$$\max(\rho_1, \rho_2) = \max(p, q), \tag{2}$$

then

$$\liminf_{r\to\infty} \frac{1}{rN(r)\phi(r)} \int_0^r \log^+ M(t, F)dt = 0,$$

where $N(r) = n(r, f) + n(r, g)$ *and* $\phi(x)$ *is any positive continuous non-decreasing function of a real variable* x *such that* $\int_a^\infty dx/x\phi(x)$ *is convergent.*

THEOREM B. *If* (1) *is a function of order* ρ, *where* $\rho > 0$ *is not an integer, then*

$$\liminf_{r\to\infty} \frac{1}{rN(r)} \int_0^r \log^+ M(t, F)dt < \infty. \tag{4}$$

In this paper we extend Theorems A and B. Let

$$F(z) = z^k \exp(H(z)) f(z)/g(z) \tag{5}$$

be any meromorphic function of finite order ρ. Here $H(z)$ is a polynomial of degree h; $f(z)$ and $g(z)$ are canonical products of orders ρ_1, ρ_2 and genera p, q respectively. The genus of $F(z)$ is $P = \max(p, q, h)$ and we have $\rho - 1 \leqq P \leqq \rho$. Let $n(r, 0)$ and $n(r, \infty)$ denote the number of zeros and poles respectively of $F(z)$ in $|z| \leqq r$ and write $\psi(r) = n(r, f) + n(r, g)$,

$$I(r, F) = I(r) = \frac{1}{r} \int_0^r \log^+ M(t, F)dt.$$

THEOREM 1. *If for any meromorphic function* (5) *of order* ρ *where*

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[1] Numbers in brackets refer to the references at the end of the paper.

$\rho>0$ is an integer,

$$h \leqq \max(p, q) = s \text{ (say)}, \tag{6}$$

then

$$\liminf_{r\to\infty} I(r, F)/\{n(r, 0) + n(r, \infty)\}\phi(r) = 0, \tag{7}$$

where $\phi(x)$ has been defined in the statement of Theorem A.

THEOREM 2. *For any meromorphic function* (5) *of order ρ where $\rho>0$ is not an integer, we have*

$$\liminf_{r\to\infty} I(r, F)/\{n(r, 0) + n(r, \infty)\} < \infty. \tag{8}$$

THEOREM 3. *If the meromorphic function* (5) *be nonconstant and of zero order, then*

$$\liminf_{r\to\infty} I(r, F)/\{N(r, 0) + N(r, \infty)\} < \infty, \tag{9}$$

where $N(r, a)$ denotes as usual

$$\int_0^r \frac{n(x, a) - n(0, a)}{x}\,dx + n(0, a)\log r.$$

COROLLARY.

$$\liminf_{r\to\infty} I(r, F)/\{n(r, 0) + n(r, \infty)\}\log r < \infty. \tag{10}$$

2. **Examples.** If $h>\max(p, q)$, then (7) does not hold. For instance, if

$$F(z) = e^z \prod_2^\infty \left\{1 + \frac{z}{n(\log n)^\alpha}\right\}, \qquad \alpha > 1,$$

then $h=1$, $\max p, q)=0$, and

$$\frac{I(r)}{\{n(r, 0) + n(r, \infty)\}\log r(\log\log r)^2} \to \infty,$$

as $r\to\infty$. Further let $\alpha(x)$ be any given function tending to infinity, however slowly, with x and consider

$$F(z) = \prod_2^\infty E\left(\frac{z}{\alpha_\nu}, p\right),$$

where

$$\alpha_\nu = -\left\{\nu(\log \nu)^\alpha\right\}^{1/\rho}, \qquad p < \rho < p + 1;$$

then $F(z)$ is an entire function of nonintegral order $\rho > 0$ and we have [5, p. 44]

$$\lim_{r\to\infty} \frac{I(r, F)\alpha(r)}{\left\{n(r, 0) + n(r, \infty)\right\}} = \infty.$$

Further $F(z) = \prod_{n=1}^{\infty} (1 - z/e^n)$ is an entire function of zero order for which

$$\lim_{r\to\infty} \frac{I(r, F)\alpha(r)}{\left\{N(r, 0) + N(r, \infty)\right\}} = \infty,$$

$$\lim_{r\to\infty} \frac{I(r, F)}{\left\{n(r, 0) + n(r, \infty)\right\} \log r} = \frac{1}{2}.$$

3. Lemma. *Let*

$$J(r, p) = \int_0^\infty \frac{r^{p+1}\psi(t)dt}{t^{p+1}(t + r)}.$$

If $s \geqq h$ *and* $\psi(t) \geqq 1$ *for all large* t, *then*[2]

(11) $$I(r, F) < HJ(r, s).$$

Proof. Let $(a_n)_1^\infty$ denote the zeros of $f(z)$ and $(b_n)_1^\infty$ the zeros of $g(z)$ and let $k > 1$. Then

$$T(r, F) < T(r, f) + T(r, g) + O(r^h + \log r).$$

$$\begin{aligned} I(r, F) &< HT(kr, F) < H\left\{T(kr, f) + T(kr, g)\right\} + O(r^h + \log r) \\ &< H\left\{\log^+ M(kr, f) + \log^+ M(kr, g)\right\} + O(r^h + \log r) \\ &< H\left\{\sum_{n=1}^{\infty} \frac{r^{s+1}}{|a_n|^s(r + |a_n|)} + \sum_1^{\infty} \frac{r^{s+1}}{|b_n|^s(r + |b_n|)}\right\} \\ &\quad + O(r^h + \log r) \\ &< H\int_0^\infty \frac{r^{s+1}\psi(t)dt}{t^{s+1}(t + r)} + O(r^h + \log r). \end{aligned}$$

Now

$$J(r, s) > \frac{1}{2}\int_0^r \frac{r^s\psi(t)dt}{t^{s+1}} > h_1 r^s.$$

Hence if $s > 0$, $I(r) < HJ(r, s)$. If $s = 0$ then $h = 0$ and since $\psi(t) \geqq 1$ for all large t,

[2] $H(h_1)$ denotes a positive constant not necessarily the same at each occurrence.

$$\int_0^r \frac{\psi(t)}{t} dt > h_1 \log r,$$

which proves the lemma.

4. **Proof of Theorem 1.** We note that $\psi(r) \geqq 1$ for all large r, for if $\psi(r) = 0$ for all r, then $s = 0$ and hence, by (6), $h = 0$ and $F(z)$ would then not be a function of order greater than or equal to one.

Consider $G(z) = \prod_1^\infty E(z/c_n, s)$, where the sequence $c_1, c_2, \cdots$ is composed of $a_1, a_2, \cdots; b_1, b_2, \cdots$ and $|c_1| \leqq |c_2| \leqq \cdots$. Since $h \leqq s$, $G(z)$ is an entire function of order ρ and genus $s = \max(p, q)$. Further $s = \rho$ or $\rho - 1$ and hence we have [2, pp. 23–29; 3, pp. 180–186]

$$\liminf_{r \to \infty} \frac{J(r, s)}{\psi(r)\phi(r)} = 0.$$

Since

$$\psi(r) \leqq n(r, 0) + n(r, \infty),$$

(7) follows from the lemma.

5. **Proof of Theorem 2.** This theorem follows from the argument of Okada [1, p. 249]. We sketch an alternative proof. Let $[\rho] = P$. Then $h \leqq P = s$. Let $0 < \epsilon < \min\{\rho - s, s + 1 - \rho\}$. From the lemma we have

$$I(r, F) < H\left\{\int_0^r \frac{r^s\psi(t)dt}{t^{s+1}} + r^{s+1}\int_r^\infty \frac{\psi(t)dt}{t^{s+2}}\right\}.$$

From Lemma 3 [3, p. 184] we have

$$\frac{\psi(t)}{t^{\rho-\epsilon}} \leqq \frac{\psi(r_n)}{r_n^{\rho-\epsilon}}, \quad 0 \leqq t \leqq r_n; \qquad \frac{\psi(t)}{t^{\rho+\epsilon}} \leqq \frac{\psi(r_n)}{r_n^{\rho+\epsilon}}, \quad t \geqq r_n,$$

for a sequence $(r_n)_1^\infty$, $r_n \uparrow \infty$. Hence the theorem follows.

6. **Proof of Theorem 3.** If $\psi(t) = 0$ for all t then $F(z)$ would be of the form Az^k and (9) and (10) obviously hold. Hence we may suppose that $\psi(t) \geqq 1$ for all large t. Let

$$\psi_1(t) = \int_0^r \frac{\psi(t)}{t} dt.$$

Then

$$I(r) < H\left[\int_0^r \frac{\psi(t)}{t} dt + r\int_r^\infty \frac{\psi(t)}{t^2} dt\right] = Hr\int_r^\infty \frac{\psi_1(t)}{t^2} dt.$$

Now

$$\lim_{r\to\infty} \frac{\psi_1(r)}{r^\epsilon} = 0.$$

Hence there exists a sequence $\{r_n\}_1^\infty$, $r_n \uparrow \infty$, such that

$$\frac{\psi_1(r)}{r^\epsilon} \leqq \frac{\psi_1(r_n)}{r_n^\epsilon} \qquad \text{for } r \geqq r_n;$$

and the theorem follows. The corollary follows directly from the theorem.

References

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Aligarh University