SOME THEOREMS ON MEROMORPHIC FUNCTIONS

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1. Introduction. In a recent paper [1] Yoshitomo Okada proved the following two theorems.

**Theorem A.** If for any meromorphic function

\[ F(z) = f(z)/g(z), \]

where \( f \) and \( g \) are canonical products of genera \( p, q \) and of orders \( p_1, p_2 \) respectively,

\[ \max (p_1, p_2) = \max (p, q), \]

then

\[ \liminf_{r \to \infty} \frac{1}{r N(r) \phi(r)} \int_0^r \log^+ M(t, F) dt = 0, \]

where \( N(r) = n(r, f) + n(r, g) \) and \( \phi(x) \) is any positive continuous non-decreasing function of a real variable \( x \) such that \( \int_0^\infty dx/x \phi(x) \) is convergent.

**Theorem B.** If (1) is a function of order \( \rho \), where \( \rho > 0 \) is not an integer, then

\[ \liminf_{r \to \infty} \frac{1}{r N(r)} \int_0^r \log^+ M(t, F) dt < \infty. \]

In this paper we extend Theorems A and B. Let

\[ F(z) = z^k \exp (H(z)) f(z)/g(z) \]

be any meromorphic function of finite order \( \rho \). Here \( H(z) \) is a polynomial of degree \( h \); \( f(z) \) and \( g(z) \) are canonical products of orders \( p_1, p_2 \) and genera \( p, q \) respectively. The genus of \( F(z) \) is \( P = \max (p, q, h) \) and we have \( \rho - 1 \leq P \leq \rho \). Let \( n(r, 0) \) and \( n(r, \infty) \) denote the number of zeros and poles respectively of \( F(z) \) in \( |z| \leq r \) and write \( \psi(r) = n(r, f) + n(r, g) \),

\[ I(r, F) = I(r) = \frac{1}{r} \int_0^r \log^+ M(t, F) dt. \]

**Theorem 1.** If for any meromorphic function (5) of order \( \rho \) where

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1 Numbers in brackets refer to the references at the end of the paper.
\( p > 0 \) is an integer,

\[ h \leq \max (p, q) = s \text{ (say)}, \]

then

\[ \liminf_{r \to \infty} \frac{I(r, F)}{n(r, 0) + n(r, \infty)} \phi(r) = 0, \]

where \( \phi(x) \) has been defined in the statement of Theorem A.

**Theorem 2.** For any meromorphic function (5) of order \( p \) where \( p > 0 \) is not an integer, we have

\[ \liminf_{r \to \infty} \frac{I(r, F)}{n(r, 0) + n(r, \infty)} < \infty. \]

**Theorem 3.** If the meromorphic function (5) be nonconstant and of zero order, then

\[ \liminf_{r \to \infty} \frac{I(r, F)}{N(r, 0) + N(r, \infty)} < \infty, \]

where \( N(r, a) \) denotes as usual

\[ \int_0^r \frac{n(x, a) - n(0, a)}{x} \, dx + n(0, a) \log r. \]

**Corollary.**

\[ \liminf_{r \to \infty} \frac{I(r, F)}{n(r, 0) + n(r, \infty)} \log r < \infty. \]

2. **Examples.** If \( k > \max (p, q) \), then (7) does not hold. For instance, if

\[ F(z) = e^z \prod_{\nu=1}^{\infty} \left( 1 + \frac{z}{n(\log n)^a} \right)^{\alpha}, \quad \alpha > 1, \]

then \( k = 1, \max (p, q) = 0, \) and

\[ \frac{I(r)}{n(r, 0) + n(r, \infty)} \log r \log r \to \infty, \]

as \( r \to \infty \). Further let \( \alpha(x) \) be any given function tending to infinity, however slowly, with \( x \) and consider

\[ F(z) = \prod_{\nu} E \left( \frac{z}{\alpha_{\nu}}, p \right), \]

where
\[ \alpha_s = -\left\{ \nu(\log \nu)^m \right\}^{1/s}, \quad \rho < \rho < \beta + 1; \]

then \( F(z) \) is an entire function of nonintegral order \( \rho > 0 \) and we have [5, p. 44]

\[ \lim_{r \to \infty} \frac{I(r, F)\alpha(r)}{\{n(r, 0) + n(r, \infty)\}} = \infty. \]

Further \( F(z) = \prod_{n=1}^{\infty} \left( 1 - z/e^n \right) \) is an entire function of zero order for which

\[ \lim_{r \to \infty} \frac{I(r, F)\alpha(r)}{\{n(r, 0) + n(r, \infty)\}} = \infty; \]

\[ \lim_{r \to \infty} \frac{I(r, F)}{\{n(r, 0) + n(r, \infty)\} \log r} = \frac{1}{2}. \]

3. Lemma. Let

\[ J(r, \beta) = \int_0^\infty \frac{r^{\beta+1} \psi(t) dt}{t^{\beta+1}(t + r)}. \]

If \( s \geq h \) and \( \psi(t) \geq 1 \) for all large \( t \), then\(^2\)

\[ I(r, F) < HJ(r, s). \]

Proof. Let \( (a_n)_1^\infty \) denote the zeros of \( f(z) \) and \( (b_n)_1^\infty \) the zeros of \( g(z) \) and let \( k > 1 \). Then

\[ T(r, F) < T(r, f) + T(r, g) + \alpha(r^k + \log r). \]

\[ I(r, F) < HT(kr, F) < H \{ T(kr, f) + T(kr, g) \} + \alpha(r^k + \log r) \]

\[ < H \{ \log^+ M(kr, f) + \log^+ M(kr, g) \} + \alpha(r^k + \log r) \]

\[ < H \left\{ \sum_{n=1}^{\infty} \frac{r^{\beta+1} \psi(t) dt}{t^{\beta+1}(t + r)} + \sum_{n=1}^{\infty} \frac{r^{\beta+1} \psi(t) dt}{t^{\beta+1}(t + r)} \right\} \]

\[ + \log^+ M(kr, f) + \log^+ M(kr, g) \]

\[ < H \int_0^\infty \frac{r^{\beta+1} \psi(t) dt}{t^{\beta+1}(t + r)} + \alpha(r^k + \log r). \]

Now

\[ J(r, s) > \frac{1}{2} \int_0^r \frac{r^s \psi(t) dt}{t^{s+1}} > h_1 r^s. \]

Hence if \( s > 0 \), \( I(r) < HJ(r, s) \). If \( s = 0 \) then \( h = 0 \) and since \( \psi(t) \geq 1 \) for all large \( t \),

\(^1 H(h) \) denotes a positive constant not necessarily the same at each occurrence.
\[ \int_0^r \frac{\psi(t)}{t} \, dt > h_1 \log r, \]
which proves the lemma.

4. **Proof of Theorem 1.** We note that \( \psi(r) \geq 1 \) for all large \( r \), for if \( \psi(r) = 0 \) for all \( r \), then \( s = 0 \) and hence, by (6), \( h = 0 \) and \( F(z) \) would then not be a function of order greater than or equal to one.

Consider \( G(z) = \prod_{n=1}^s E(z/c_n, s) \), where the sequence \( c_1, c_2, \ldots \) is composed of \( a_1, a_2, \ldots \); \( b_1, b_2, \ldots \) and \( |c_1| \leq |c_2| \leq \ldots \). Since \( h \leq s \), \( G(z) \) is an entire function of order \( p \) and genus \( s = \max (p, q) \). Further \( s = p \) or \( p - 1 \) and hence we have \([2, \text{pp. 23–29}; 3, \text{pp. 180–186}]\)

\[ \lim_{r \to \infty} \frac{J(r, s)}{\psi(r)\phi(r)} = 0. \]

Since

\[ \psi(r) \leq n(r, 0) + n(r, \infty), \]

(7) follows from the lemma.

5. **Proof of Theorem 2.** This theorem follows from the argument of Okada \([1, \text{p. 249}]\). We sketch an alternative proof. Let \( [p] = P \). Then \( h \leq P = s \). Let \( 0 < \epsilon < \min \{ p-s, s+1-p \} \). From the lemma we have

\[ I(r, F) < H \left\{ \int_0^r \frac{r^s \psi(t) \, dt}{t^{s+1}} + r^{s+1} \int_r^\infty \frac{\psi(t) \, dt}{t^{s+2}} \right\}. \]

From Lemma 3 \([3, \text{p. 184}]\) we have

\[ \frac{\psi(t)}{t^{p-s}} \leq \frac{\psi(r_n)}{r_n^{p-s}}, \quad 0 \leq t \leq r_n; \quad \frac{\psi(t)}{t^{p-s}} \leq \frac{\psi(r_n)}{r_n^{p-s}}, \quad t \geq r_n, \]

for a sequence \( (r_n)_1^\infty, r_n \uparrow \infty \). Hence the theorem follows.

6. **Proof of Theorem 3.** If \( \psi(t) = 0 \) for all \( t \) then \( F(z) \) would be of the form \( Az^k \) and (9) and (10) obviously hold. Hence we may suppose that \( \psi(t) \geq 1 \) for all large \( t \). Let

\[ \psi_1(t) = \int_t^r \frac{\psi(t)}{t} \, dt. \]

Then

\[ I(r) < H \left[ \int_0^r \frac{\psi(t)}{t} \, dt + r \int_r^\infty \frac{\psi(t)}{t^2} \, dt \right] = Hr \int_r^\infty \frac{\psi_1(t)}{t^2} \, dt. \]
Now
\[
\lim_{r \to \infty} \frac{\psi_1(r)}{r^*} = 0.
\]
Hence there exists a sequence \( \{r_n\}_{n=1}^{\infty}, r_n \uparrow \infty \), such that
\[
\frac{\psi_1(r)}{r^*} \leq \frac{\psi_1(r_n)}{r_n^*} \quad \text{for } r \geq r_n;
\]
and the theorem follows. The corollary follows directly from the theorem.

References


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