SELF-ADJOINT FACTORIZATIONS OF
DIFFERENTIAL OPERATORS

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In this short paper we prove the following result:

**Theorem.** Let \( L \) be an ordinary linear differential operator

\[
L = p_0(x) \frac{d^n}{dx^n} + p_1(x) \frac{d^{n-1}}{dx^{n-1}} + \cdots + p_n(x)\cdot
\]

of even order \( n = 2r \). \( p_i(x) \in C^{n-i} \) and \( p_0(x) > 0 \) in some closed finite interval \([a, b]\). Then there exists a subinterval of \([a, b]\) in which \( L \) has a factorization

\[
L = f(x) P_1 P_2 \cdots P_r,
\]

where each \( P_a \) is of the second order and formally self-adjoint.

The theorem follows by complete induction after the proofs of Lemmas 1 and 2 below. We use the following notation: If \( M \) is a linear differential operator, then its formal or Lagrange adjoint will be denoted by \( M^+ \).

**Lemma 1.** Let

\[
N = \frac{d^n}{dx^n} + q_1(x) \frac{d^{n-1}}{dx^{n-1}} + \cdots + q_n(x)\cdot
\]

be a linear differential operator with \( q_i(x) \in C^0 \) in some closed finite interval \([a, b]\). Then there is a subinterval \([a', b']\) of \([a, b]\) in which \( N \) has the representation

\[
N = PM
\]

where \( P = P^+ \) is of second order.

**Proof.** Let \( \{\phi_1(x), \phi_2(x), \cdots, \phi_n(x)\} \) be \( n \) linearly independent solutions of \( Nu = 0 \) with Wronskian \( W(x) \). There exist \( n - 2 \) functions among the \( \phi_1, \phi_2, \cdots, \phi_n \) whose Wronskian \( \omega(x) \) is not identically zero in some subinterval of \([a, b]\). Let these \( n - 2 \) functions be \( \phi_1(x), \phi_2(x), \cdots, \phi_{n-2}(x) \) and let \( \omega(x) \) be unequal to zero in \([a', b']\).

Define the operator \( M \) by the equation:

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\[ M u = \begin{bmatrix} +1 & \phi_1(x) & \phi_2(x) & \cdots & \phi_{n-2}(x) & u \\ W(x) & \phi_1'(x) & \phi_2'(x) & \cdots & \phi_{n-2}'(x) & u' \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \phi_1^{(n-2)}(x) & \phi_2^{(n-2)}(x) & \cdots & \phi_{n-2}^{(n-2)}(x) & u^{(n-2)} \end{bmatrix} \]

\[ = s_2(x)u^{(n-2)} + s_3(x)u^{(n-3)} + \cdots + s_n(x)u. \]

Let

\[ P = a(x) \frac{d^2}{dx^2} + b(x) \frac{d}{dx} + c(x). \]

be so chosen that \( PM = N. \) Now

\[ PMu = as_2u^{(n)} + [a(2s_2' + ss) + bs_2]u^{(n-1)} + \cdots, \]

\[ Nu = u^{(n)} + q_1u^{(n-1)} + \cdots. \]

Comparing coefficients and noting that \((Ws_2)' + (W_{s_3}) = 0,\) we see that \(a'(x) = b(x)\) and hence that \( P = P^+. \)

**Lemma 2.** Let

\[ L = \frac{d^n}{dx^n} + p_1(x) \frac{d^{n-1}}{dx^{n-1}} + \cdots + p_n(x). \]

be a linear differential operator with \( p_i(x) \in C^{n-i}, \) \( p_0(x) > 0 \) in some closed finite interval \([a, b].\) Then there is subinterval \([a', b']\) of \([a, b]\) such that \( L \) has a representation

\[ L = SQ \]

where \( Q = Q^+ \) is of second order.

**Proof.** Let \( N = (1/p_0(x))L. \) Then \( N \) is a linear differential operator with leading coefficient 1. Hence \( N^+ \) has leading coefficient 1. By Lemma 1, there exists a subinterval of \([a, b]\) such that

\[ N^+ = QR \]

with \( Q = Q^+. \) Taking adjoints of the above equation:

\[ N = R^+Q^+ = R^+Q. \]

Now

\[ L = \frac{d^n}{dx^n} = p_0(x)R^+Q. \]

Let \( p_0(x)R^+ = S. \) Then \( L = SQ. \)

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