SELF-ADJOINT FACTORIZATIONS OF DIFFERENTIAL OPERATORS

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In this short paper we prove the following result:

**Theorem.** Let $L$ be an ordinary linear differential operator

$$L = p_0(x) \frac{d^n}{dx^n} + p_1(x) \frac{d^{n-1}}{dx^{n-1}} + \cdots + p_n(x).$$

of even order $n = 2r$. $p_i(x) \in C^{n-i}$ and $p_0(x) > 0$ in some closed finite interval $[a, b]$. Then there exists a subinterval of $[a, b]$ in which $L$ has a factorization

$$L = f(x) P_1 P_2 \cdots P_r,$$

where each $P_a$ is of the second order and formally self-adjoint.

The theorem follows by complete induction after the proofs of Lemmas 1 and 2 below. We use the following notation: If $M$ is a linear differential operator, then its formal or Lagrange adjoint will be denoted by $M^+.$

**Lemma 1.** Let

$$N = \frac{d^n}{dx^n} + q_1(x) \frac{d^{n-1}}{dx^{n-1}} + \cdots + q_n(x).$$

be a linear differential operator with $q_i(x) \in C^0$ in some closed finite interval $[a, b].$ Then there is a subinterval $[a', b']$ of $[a, b]$ in which $N$ has the representation

$$N = PM$$

where $P = P^+$ is of second order.

**Proof.** Let $\{\phi_1(x), \phi_2(x), \cdots, \phi_n(x)\}$ be $n$ linearly independent solutions of $Nu = 0$ with Wronskian $W(x).$ There exist $n - 2$ functions among the $\phi_1, \phi_2, \cdots, \phi_n$ whose Wronskian $\omega(x)$ is not identically zero in some subinterval of $[a, b].$ Let these $n - 2$ functions be $\phi_1(x), \phi_2(x), \cdots, \phi_{n-2}(x)$ and let $\omega(x)$ be unequal to zero in $[a', b'].$

Define the operator $M$ by the equation:

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\[ M u = \frac{+1}{W(x)} \begin{vmatrix} \phi_1(x) & \phi_2(x) & \cdots & \phi_{n-2}(x) & u \\ \phi_1'(x) & \phi_2'(x) & \cdots & \phi_{n-2}'(x) & u' \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \phi_1^{(n-2)}(x) & \phi_2^{(n-2)}(x) & \cdots & \phi_{n-2}^{(n-2)}(x) & u^{(n-2)} \end{vmatrix} = s_2(x)u^{(n-2)} + s_3(x)u^{(n-3)} + \cdots + s_n(x)u. \]

Let

\[ P = a(x) \frac{d^2}{dx^2} + b(x) \frac{d}{dx} + c(x). \]

be so chosen that \( PM = N \). Now

\[ PMu = as_2u^{(n)} + \left[ a(2s_2' + ss) + bs_2 \right] u^{(n-1)} + \cdots, \]
\[ Nu = u^{(n)} + q_1u^{(n-1)} + \cdots. \]

Comparing coefficients and noting that \( (Ws_2)' + (Ws_3) = 0 \), we see that \( a'(x) = b(x) \) and hence that \( P = P^+ \).

**Lemma 2.** Let

\[ L = \sum_{i=0}^{n} p_i(x) \frac{d^i}{dx^i} \]

be a linear differential operator with \( p_i(x) \in C^{n-i} \), \( p_0(x) > 0 \) in some closed finite interval \([a, b]\). Then there is a subinterval \([a', b']\) of \([a, b]\) such that \( L \) has a representation

\[ L = SQ \]

where \( Q = Q^+ \) is of second order.

**Proof.** Let \( N = (1/p_0(x))L \). Then \( N \) is a linear differential operator with leading coefficient 1. Hence \( N^+ \) has leading coefficient 1. By Lemma 1, there exists a subinterval of \([a, b]\) such that

\[ N^+ = QR \]

with \( Q = Q^+ \). Taking adjoints of the above equation:

\[ N = R^+Q^+ = R^+Q. \]

Now

\[ L = p_0(x)N = p_0(x)R^+Q. \]

Let \( p_0(x)R^+ = S \). Then \( L = SQ \).

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