THE WEDDERBURN PRINCIPAL THEOREM
IN BANACH ALGEBRAS

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The Principal Theorem of Wedderburn for a finite-dimensional algebra $A$ states that $A$ is the vector space direct sum of its radical $R$ and an algebra isomorphic to $A/R$. It will be shown that the corresponding theorem is not true for all Banach algebras, but that it is true with certain restrictions.

The terminology of Jacobson [3] will be followed for radical, quasi-inverse, and quasi-regular. The notations $x \circ y = x + y + xy$ and $x'$ for the quasi-inverse of $x$ will also be employed.

**Definition 1.** A Banach algebra is a complete normed linear space which is also an algebra over the complex numbers satisfying $\|xy\| \leq \|x\| \|y\|$.

All the following results are proved for real algebras in [1] by the same methods.

To show that the Wedderburn theorem does not hold for an arbitrary Banach algebra, consider the commutative algebra $A$ which is the completion of the algebra of all finite sums

$$\sum_{i=1}^{n} \alpha_i e_i + \beta r$$

where $\alpha_i$ and $\beta$ are complex, $e_i$ are mutually orthogonal idempotents, $r^2 = 0$, $e_i r = r e_i = 0$, and

$$\|\sum \alpha_i e_i + \beta r\| = \max \{ \|\sum |\alpha_i|^2|^{1/2}, |\beta - \sum \alpha_i| \}.$$  

It is easy to show this defines a norm, but it is also necessary to verify that $\|xy\| \leq \|x\| \|y\|$. Let $x = \sum \alpha_i e_i + \gamma r$, $y = \sum \beta e_i e_i + \nu r$. Then $xy = \sum \alpha_i \beta_i e_i$, $\|xy\| = \max \{ [\sum |\alpha_i \beta_i|^2]^{1/2}, [\sum |\alpha_i|^2]^{1/2} \}$. By the Cauchy inequality,

$$|\sum \alpha_i \beta_i| \leq \sum |\alpha_i \beta_i| \leq [\sum |\alpha_i|^2]^{1/2} [\sum |\beta_i|^2]^{1/2}.$$  

Together with $\sum |\alpha_i \beta_i|^2 \leq \sum |\alpha_i|^2 \sum |\beta_i|^2$ this shows $\|xy\| \leq \|x\| \|y\|$. Hence $A$ is a Banach algebra.

$A/R$ is the algebra of all sequences $\sum_{i=1}^{n} \alpha_i u_i$ where $u_i^2 = u_i$, $\alpha_i$ are complex, and $\|\sum \alpha_i u_i\| = [\sum |\alpha_i|^2]^{1/2} < \infty$. $A/R$ contains the element $x = \sum_{i=1}^{n} t^{-1} u_i$, since $\sum t^{-2} u_i = \pi_i^2/6$, but there is no element

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1 Numbers in brackets refer to the references cited at the end of the paper.
\[
\sum i^{-1} e_i \text{ in } A, \text{ for } \sum i^{-1} \text{ diverges. Therefore there is no subalgebra of } A \text{ isomorphic to } A/R.
\]

It can be shown [1] that the radical of } A \text{ is one-dimensional. Thus no restriction on the dimension of the radical will suffice. However it will now be shown that it is sufficient for } A/R \text{ to be finite-dimensional.}

**Theorem 1.** If } A \text{ is a Banach algebra, } R \text{ its radical, and } A/R \text{ is finite-dimensional, then there is a subalgebra } S \text{ of } A \text{ isomorphic and homeomorphic to } A/R. \text{ } A \text{ is the vector space direct sum } S+R.

**Lemma 1.** If } A \text{ is a Banach algebra, } R \text{ its radical, and } \{u_i\} \text{ a denumerable set of pairwise orthogonal idempotents of } A/R, \text{ then there exist idempotents } e_i \text{ in } A \text{ mapping on } u_i \text{ via } A \rightarrow A/R, \text{ and the } e_i \text{ are pairwise orthogonal.}

The proof is by induction. Let } a_1 \text{ be an element of } A \text{ mapping on the class } u_1. \text{ Then } a_1^2 - a_1 = r_1 \text{ in } R \text{ by hypothesis. For any } r \text{ in } R \text{ there exists } (1+4r)^{-1/2} = 1 - 2r + 6r^2 - 20r^3 + \cdots \text{ since } \|r^n\|^{1/n} \rightarrow 0 \text{ [2] guarantees convergence of this series. Define } e_1 = (2a_1 - 1) [2(1+4r)^{-1/2}]^{-1} + 1/2 = a_1(1 - 2r + 6r^2 - \cdots) + r - 3r^2 + 10r^3 - \cdots. \text{ Then } e_i^2 = e_i \text{ and } e_i \text{ maps on } u_i \text{ since } a_1 \text{ does. Assume there exist } e_1, \cdots, e_{t-1} \text{ such that } e_i = e_i, e_i e_j = 0 = e_j e_i \text{ for } i \neq j, \text{ and } e_i \rightarrow u_i, \text{ } i = 1, 2, \cdots, t-1. \text{ Define } f = \sum e_i e_i. \text{ Then } f^2 = f, \text{ } e_i f = f e_i. \text{ Let } b_i \text{ be any element such that } b_i \rightarrow u_i. \text{ Define } a_i = (1 - f) b_i(1 - f). \text{ Then } a_i a_i = a_i e_i = 0, \text{ and } a_i u_i, \text{ since } f a_i = 0, \text{ and } fb_i = 0. \text{ Hence } a_i^2 - a_i = r_i \text{ in } R \text{ and } e_i r_i = r_i e_i = 0, \text{ } i = 1, 2, \cdots, t-1. \text{ Define } e_i = (2a_i - 1) [2(1+4r)^{-1/2}]^{-1} + 1/2. \text{ Then } e_i = e_i, \text{ } e_i \rightarrow u_i, \text{ and } e_i e_i = e_i e_i = 0 \text{ since } e_i a_i = e_i r_i = 0. \text{ This completes Lemma 1.}

**Lemma 2.** If } A/R \text{ contains a ring direct sum } M_1 \oplus M_2 \oplus \cdots \oplus M_t \text{ of total matrix algebras } M_i, \text{ then } A \text{ contains a ring direct sum of total matrix algebras } S_i \rightarrow M_i \text{ via } A \rightarrow A/R.

Consider first a single matrix algebra } M \subset A/R, \text{ where } M \text{ is generated over the complexes by } u_{ij}, \text{ } u_{ii} \text{ are pairwise orthogonal idempotents, } u_{ij} u_{jk} = u_{ik}, \text{ and } u_{ij} u_{ik} = 0 \text{ for } k \neq j. \text{ Since there are a finite number of } u_{ii}, \text{ by Lemma 1 } A \text{ contains idempotents } e_{ii} \rightarrow u_{ii} \text{ with } e_{ii} e_{jj} = e_{jj} e_{ii} = 0 \text{ for } i \neq j. \text{ Choose an element } v_{ii} \rightarrow u_{ii} \text{ and an element } v_{jj} \rightarrow u_{jj}. \text{ Since } u_{ii} u_{il} u_{il} = u_{ii} \text{ and } u_{ii} u_{ij} u_{jj} = u_{ij}, \text{ } v_{ii} \text{ may be chosen in } e_{ii} A e_{ii}; \text{ } v_{ij} \text{ may be chosen in } e_{ij} A e_{jj}. \text{ Then } v_{ij} v_{ji} = u_{ij}, \text{ and } v_{ij} v_{ji} = u_{ji}. \text{ Hence } v_{ij} v_{ji} = v_{ij} + a_j \text{ where } a_j \text{ is in } R \cap e_{ij} A e_{ii}. \text{ By [3], } a_j \text{ exists. } (e_{ii} + a_j)(e_{ii} + a_j) = e_{ii} + a_j e_{ii} + e_{ii} a_j + a_j a_j = e_{ii} \text{ since } a_j = \sum (-a_j)^n = 0 \text{ also in } e_{ii} A e_{ii}. \text{ Define } e_{ij} = e_{ii} e_{ij}. \text{ Then } e_{ij} e_{jk} = e_{ik} \text{ and } e_{ij} e_{jk} = 0 \text{ for } j \neq k. \text{ Clearly } e_{ij} is
in $A$ and $e_{ij} ightarrow u_{ij}$. Thus $A$ contains a total matric algebra $(e_{ij})$ isomorphic to $M$. The sum of the algebras $S_i$ so constructed for each $M_i$ is the ring direct sum since the basis elements are constructed from mutually orthogonal idempotents. This completes Lemma 2.

Proof of Theorem. $A/R$ is the direct sum of a finite number of finite-dimensional total matric algebras over the complex numbers. Hence $A$ contains a subalgebra $S \cong A/R$. Since the isomorphism $S \rightarrow A/R$ is continuous, it is a homeomorphism. $S$ is semi-simple; so $S \cap R = 0$. Therefore $S + R$ is a vector space direct sum.

When $A/R$ is not finite-dimensional the theorem can still be proved if $R$ is finite-dimensional and $A/R$ is a well known type of algebra most generally defined in [4] as follows:

Definition 2. The $B(\infty)$ direct sum of a denumerable number of algebras $A_i$ is the completion in a specified norm of the algebra of all sequences $\{a_i\}$ such that $a_i$ in $A_i$ are 0 for all but a finite number of $i$.

Theorem 2. If $A$ is a Banach algebra, the radical $R$ of $A$ is finite-dimensional, and $A/R$ is the $B(\infty)$ direct sum of finite-dimensional total matric algebras, then $A$ is a vector space direct sum, $A = B + C + D$, where $B$ is finite-dimensional, $BC = CB = 0$, every idempotent of $C$ mapping on an element in the basis of $A/R$ is orthogonal to $R$, and $D \subset R$. When $A$ is commutative, $D = 0$ and $A$ is a ring direct sum of $B$ and $C$.

Let $n$ be the dimension of $R$. Then there are at most $n$ distinct primitive orthogonal idempotents $e_k$ and $n$ distinct primitive orthogonal idempotents $e_s$ of $A$ for which $e_k e_s \neq 0$ and $r e_s \neq 0$ for any $r_k$ and $r_s$ in $R$. Otherwise

$$e_{n+1} r_{n+1} = \sum_{k=1}^{n} a_k e_{n+1}$$

for complex $a_k$ and $\beta_s$, since any $n+1$ elements of $R$ are linearly dependent. However,

$$e_{n+1}(e_{n+1} r_{n+1}) = e_{n+1} r_{n+1} = \sum_{k=1}^{n} a_k e_{n+1} e_k r_k = 0,$$

$$(r_{n+1} e_{n+1})e_{n+1} = r_{n+1} e_{n+1} = \sum_{k=1}^{n} \beta_k e_k e_{n+1} = 0.$$

Hence there are at most $2n$ primitive orthogonal idempotents $e_j$ for which $e_j R \neq 0$ or $Re_j \neq 0$.

Let $\{u_{ij}\}$ be a basis for the matric algebras of $A/R$. Choose a fixed set of $e_{ij}$ constructed as in Lemma 2 to map on $u_{ij}$, and number the set so that $e_j = e_{jj}$, $j = 1, \ldots, s$, are all idempotents of the set $\{e_{ij}\}$ which are not orthogonal to the radical. Define $e = \sum_{j=1}^{s} e_j$, $B = eAe$, $C = (1 - e)A(1 - e)$, and $D = eA(1 - e) + (1 - e)Ae$. Then $A = B + C + D$.
is the usual two-sided Peirce decomposition of $A$. Obviously $BC = CB = 0$.

If $A$ is commutative, $e(1-e) = 0$; so $D = 0$. Therefore $A$ is a ring direct sum, $A = B \oplus C$.

Note that if $e_i = e_{ii}$ is an idempotent of $\{e_{ij}\}$ which is orthogonal to $R$ and $e_k = e_{kk}$ is an idempotent of $\{e_{ij}\}$ which maps on $u_k = u_{kk}$ in the same matric algebra as $u_{ii}$, then $e_{kk}$ is also orthogonal to $R$, since by Lemma 2 there exist $e_{ik}$ and $e_{ki}$ such that $e_{ik}e_{ii}e_{ik} = e_{kk}$. Then $e_{kk}R = e_{kk}e_{ii}e_{kk}R = 0$, and $Re_{kk} = Re_{kk}e_{kk} = 0$.

Let $u$ be the image of $e$ under $A \rightarrow A/R$. Then $u$ is the sum $u = I_1 + \cdots + I_n$ where $I_m$ is the unit element of a matric algebra in $A/R$. Now $D = u(A/R)(1-u) + (1-u)(A/R)u$. Since $u$ commutes with $A/R$, $D \rightarrow 0$. Therefore $DCR$. $eAe/R$ is finite-dimensional and $R$ is finite-dimensional. Therefore $eAe$ is finite-dimensional. All idempotents of $\{e_{ij}\}$ not orthogonal to $R$ are in $B$; so all idempotents of $\{e_{ij}\}$ in $C$ are orthogonal to $R$. This completes Theorem 2.

The Principal Theorem of Wedderburn is known for finite-dimensional algebras, so $B = S_1 + R_1$. If it can be proved that $C = S_2 + R_2$, then it is proved for $A$; for $S = S_1 + S_2$ is a subalgebra, and it follows from $BC = CB = 0$ that $S_1S_2 = S_2S_1 = 0$, which implies $S_1 \cong A/R$.

A $C^*$-algebra is a Banach algebra with a conjugate linear involution $x \rightarrow x^*$ such that $(xx^*)^n$ exists for all $x$ and $\|xx^*\| = \|x\|^2$. It is proved in [4] that a completely continuous $C^*$-algebra is the $B(\infty)$ direct sum of finite-dimensional total matric algebras.

**Theorem 3.** If $A/R$ is a completely continuous $C^*$-algebra and $R$ is finite-dimensional, then $A$ is a vector space direct sum, $A = S + R$, of $R$ and an algebra $S$ isomorphic and homeomorphic to $A/R$.

Theorem 2 applies to give $A = B + C + D$. The remark above implies a continuous isomorphism between $S_1$ and $B/R_1$. By the closed graph theorem this is a homeomorphism; so it remains to prove the theorem only for the algebra $C$ in which every idempotent of the set $\{e_{ij}\}$ is orthogonal to $R$. It will thus be assumed that all idempotents in the set $\{e_{ij}\}$ are orthogonal to $R$.

**Lemma 3.** All elements of $\{e_{ij}\}$ are orthogonal to $R$.

Since $e_{ij} = e_{ii}e_{ij} = e_{ij}e_{jj}$, and it has been assumed that all idempotents are orthogonal to $R$, it is clear that all $e_{ij}$ are.

**Lemma 4.** $\|e_{ij}\| = \|u_{ij}\| = 1$.

By [5, Theorem 10] and [4] the basis $\{u_{ij}\}$ may be chosen so that $u_{ij}^* = u_{ji}$.
\[ \left\| u_{ij} u_{ij}^* \right\| = \left\| u_{ij} \right\| = \left\| u_{ij} \right\|^2. \] Hence \( \left\| u_{ij} \right\| = 1. \)
\[ \left\| u_{ij} u_{ij}^* \right\| = \left\| u_{ij} \right\| = \left\| u_{ij} \right\|^2. \] Hence \( \left\| u_{ij} \right\| = 1. \) By definition,
\[ \inf_{r \in \mathbb{R}} \left\| e_{ij} + r \right\| = \left\| u_{ij} \right\| = 1. \]

Let \( n \) be the dimension of \( R. \) Then \( r^{n+1} = 0. \)
\[ \left\| (e_{ii} + r)^{(n+1)} \right\| = \left\| e_{ii}^{n+1} + (n + 1) e_{ii} r + \cdots + r^{n+1} \right\| \]
\[ = \left\| e_{ii} \right\| \leq \left\| e_{ii} + r \right\|^{n+1}. \]

For any \( \varepsilon > 0 \) there is an \( r \) in \( R \) for which \( \left\| e_{ii} + r \right\| ^* < 1 + \varepsilon. \) Hence \( \left\| e_{ii} \right\| \leq 1. \) Since \( \left\| e_{ii} \right\| \leq \left\| e_{ii} \right\|^2, \) \( \left\| e_{ii} \right\| = 1. \) Now
\[ \inf_{r \in \mathbb{R}} \left\| e_{ii} + r \right\| = \left\| u_{ii} \right\| = 1, \]
\[ \inf_{r \in \mathbb{R}} \left\| e_{ij} + r \right\| = \left\| u_{ij} \right\| = 1, \]
\[ e_{ii} = (e_{ii} + r)e_{ii}, \]
\[ \left\| e_{ii} \right\| \leq \left\| e_{ii} + r \right\| \left\| e_{ii} \right\| = \left\| e_{ii} + r \right\|, \]
\[ e_{ij} = e_{ii}(e_{ij} + r), \]
\[ \left\| e_{ij} \right\| \leq \left\| e_{ij} \right\| \left\| e_{ij} + r \right\| = \left\| e_{ij} + r \right\|. \]

This shows \( \left\| e_{ij} \right\| \leq 1 \) and \( \left\| e_{ij} \right\| \leq 1. \) The mapping \( e_{ij} \mapsto u_{ij} \) depresses the norm.
\[ \left\| e_{ij} \right\| \leq \left\| e_{ij} \right\| \left\| e_{ij} \right\| \leq 1. \]

Therefore \( \left\| e_{ij} \right\| = 1. \)

**Proof of theorem.** \( A/R \) is the \( B(\infty) \) direct sum of finite-dimensional total matrix algebras \( M_i. \) By Lemma 4, \( A \) contains a subalgebra \( S_i \) equivalent to \( M_i. \) It will be shown that the map of any finite sum \( \sum_{i=1}^{k} N_i, \) \( N_i \) in \( S_i, \) into \( A/R \) is an isometry. Suppose \( N_i \rightarrow N_i \) in \( M_i, \) and \( I_i \) is the identity matrix of \( S_i. \) Since \( A/R \) is a \( C^* \) algebra, \( (I_i I_i^*)^* = I_i = I_i I_i^*, \) \( \left\| I_i \right\| = \left\| I_i I_i^* \right\| = \left\| I_i \right\|^2; \) so \( \left\| I_i \right\| = 1. \) Furthermore
\[ \left\| I_1 + \cdots + I_i \right\| = \left\| (I_1 + \cdots + I_i)(I_i^* + \cdots + I_i^*) \right\|^*, \]
\[ \left\| I_1 + \cdots + I_i \right\| = \left\| I_1 + \cdots + I_i \right\|^2 = 1. \]

Define \( I = I_1 + \cdots + I_k. \) Then
\[ \inf_{r \in \mathbb{R}} \left\| I + r \right\| = \left\| I \right\| = 1. \]
\[ \left\| (I + r)^{n+1} \right\| = \left\| I \right\| \leq \left\| I + r \right\|^{n+1}. \]

Hence \( \left\| I \right\| = 1, \) and similarly
\[\inf_{r \in R} \left\| \sum_{i=1}^t N_i + r \right\| = \left\| \sum N_i \right\|,\]

\[I(\sum N_i + r) = \sum N_i,\]

\[\left\| \sum N_i \right\| \leq \left\| I \right\| \left\| \sum N_i + r \right\| = \left\| \sum N_i + r \right\|,\]

\[\left\| \sum N_i \right\| \leq \left\| \sum N_i \right\|.

Since the mapping \(A \rightarrow A/R\) depresses norms,

\[\left\| \sum N_i \right\| = \left\| \sum N_i \right\|.

This shows that the mapping of any finite sum \(\sum_{i=1}^t N_i\) into \(A/R\) is an isometry. Let \(S\) be the \(B(\infty)\) direct sum of the subalgebras \(S_i\) of \(A\). Since \(A\) is complete, \(S \subseteq A\). A dense subset of \(S\) maps isometrically and isomorphically onto a dense subset of \(A/R\); therefore \(S\) is isomorphic and isometric to \(A/R\). This proves Theorem 3.

The theorem will now be proved for an algebra in which the mapping \(A \rightarrow A/R\) depresses the norm as little as possible.

**Definition 3.** An \(l_1\) algebra is the commutative Banach algebra of all sums \(\sum \alpha_i u_i\), where \(\alpha_i\) are complex, \(u_i\) are a denumerable number of primitive orthogonal idempotents, and \(\sum \left\| \alpha_i u_i \right\| = \sum \left| \alpha_i \right| < \infty\).

**Theorem 4.** If \(A/R\) is an \(l_1\) algebra and \(R\) is finite-dimensional, then \(A = S + R\) where \(S\) is a subalgebra of \(A\) isomorphic and homeomorphic to \(A/R\).

As in Theorem 3 it is sufficient to consider an algebra \(A\) in which each idempotent \(e_i\) is orthogonal to \(R\).

There exist pairwise orthogonal idempotents \(e_i \rightarrow u_i\) by Lemma 1. The proof of Lemma 4 shows \(\left\| e_i \right\| = 1\). For any \(x = \sum \alpha_i e_i\) in \(A\),

\[\left\| x \right\| \leq \left\| \sum \alpha_i e_i \right\| = \sum \left| \alpha_i \right| \left\| e_i \right\| = \sum \left| \alpha_i \right| = \left\| \sum \alpha_i u_i \right\|,\]

and the mapping \(A \rightarrow A/R\) decreases norms. Hence \(\left\| x \right\| = \sum \left| \alpha_i \right|\), that is, the mapping is an isometry on the completion \(S\) of the subalgebra of \(A\) generated by the \(e_i\). Therefore \(S\) is isometric and isomorphic to \(A/R\) and \(A = S + R\). This completes the proof.

In all the previous theorems the completion of the algebra generated by elements mapping on basis elements of \(A/R\) is disjoint from the radical. The following theorem shows this property is the essential one.

**Theorem 5.** Suppose \(A\) is a Banach algebra, that the radical \(R\) is finite-dimensional, that \(A/R\) is the \(B(\infty)\) sum of finite-dimensional total matric algebras, that \(S\) is the \(B(\infty)\) sum in \(A\) of the matric algebras isomorphic to those of \(A/R\), and that \(S \cap R = 0\). Then \(S\) is isomorphic and homeomorphic to \(A/R\), and \(A\) is the vector space direct sum \(S + R\).
$S$ is complete and $R$ is complete since the radical of a Banach algebra is closed. $R$ is finite-dimensional so $S+R$ is complete. Also $(S+R)/R$ is complete; hence $A\rightarrow A/R$ maps $S+R$ onto $A/R$. $S\cap R = 0$ implies $(S+R)/R = S$. Therefore $S\cong A/R$. The mapping $S\rightarrow A/R$ is 1-1 and continuous. By the closed graph theorem, $S$ is homeomorphic to $A/R$. Suppose $a$ in $A$ maps on $[a]$ in $A/R$. Then there is an $s$ in $S$ which maps on $[a]$. Thus $a-s=r$ in $R$. Every $a=s+r$. Since $S$ is semi-simple, $A=S+R$.

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