NOTE ON THE LOCATION OF THE CRITICAL POINTS
OF A REAL RATIONAL FUNCTION

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The object of this note is to study the critical points of a real rational function \( p(z) \) of degree \(-2m\) (>0) with \(-2m\) poles in the closed interior of the unit circle \( C: |z| = 1 \), with a real zero of multiplicity \( k\) (>0) in the point \( x_i \) interior to \( C \), and with a zero of multiplicity \(-2m-k\) (>0) at infinity. The corresponding problems for the case \( |x_i| > 1 \), and for the case that \( p(z) \) is of degree \( k \) with infinity no longer a zero but a pole of \( p(z) \) of multiplicity \( k+2m\) (>0) with no restrictions on \( |x_i| \), have already been treated elsewhere;\(^1\) we retain the notation and terminology of that previous treatment.

We shall prove

**Lemma 5.** Let \( C: |z| = 1 \) be the unit circle, let \( \lambda \ (>1) \) be constant, and let \( A: z = x_i \) be a real point interior to \( C \). If \( P \) is a variable nonreal point, we denote by \( Q \) the intersection other than \( P \) of the line \( AP \) with the circle through \(-1, +1, \) and \( P \). Then the locus of points \( P: (x, y) \) such that we have \( QP/AP = \lambda \) consists of the nonreal points of the circle

\[
(\lambda - 1)[(x - x_i)^2 + y^2] + x_i^2 - 1 = 0.
\]

If \( P \) is the point \((x, y)\), then the circle through \(-1, +1, \) and \( P \) is

\[
X^2 + [Y - (x^2 + y^2 - 1)/2y]^2 = 1 + (x^2 + y^2 - 1)^2/4y^2,
\]

where the running coordinates are \( X \) and \( Y \). The point \( Q \) has the coordinates \((\lambda x + x - \lambda x, y - \lambda y)\), and a necessary and sufficient condition that \( Q \) lie on the circle (10) is precisely (9). Equation (9) represents a proper circle.

**Lemma 6.** Let \( C: |z| = 1 \) be the unit circle, and \( A: z = x_i \) a real point interior to \( C \). If the point \( P: (x, y) \) lies exterior to the circle (9), then for every point \( Q' \) collinear with \( A \) and \( P \), separated by \( A \) from \( P \), and lying interior to the circle (10), we have \( Q'P/AP < \lambda \).

It is to be noted that \( A \) is the center of the circle (9). When \( P \) moves on \( AP \) monotonically away from \( A \), the point \( Q \) of Lemma 5 moves monotonically toward \( A \). Consequently the ratio \( QA/AP \)

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\(^1\) J. L. Walsh, *The location of critical points*, Amer. Math. Soc. Colloquium Publications, vol. 34, §5.8.3. All references in the present note are to that book.
$= QP/AP - 1$ decreases monotonically, and Lemma 6 follows.

We are now in a position to establish our main result:

**Theorem 8.** Let $p(z)$ be a rational function of degree $-2m$ all of whose poles lie in the closed interior of the unit circle $C$, which has a zero of multiplicity $k$ $(> 0)$ in the real point $x_1$ interior to $C$, and which has a zero of multiplicity $-2m - k$ $(> 0)$ at infinity. Then all nonreal finite critical points of $p(z)$ exterior to $C$ lie in the closed interior of the circle

$$(k + 2m)[(x - x_1)^2 + y^2] + k(1 - x_1) = 0.$$  

Let $P$ be a nonreal critical point of $p(z)$ exterior to $C$, hence a position of equilibrium in the usual field of force. The particles at the poles of $p(z)$ total $2m$ in mass, are symmetric in the axis of reals, and lie in the closed interior of $C$; hence, by the method of proof of Lemma 2, the force these particles exert at $P$ is equal to the force exerted at $P$ by a single particle of mass $2m$ located at some point $Q'$ of $R$ (notation of Lemma 2). Then $P$ is a position of equilibrium in the field due to this particle at $Q'$, and to a particle of mass $k$ in the point $A : z = x_1$. Consequently $Q'$ lies in $R_0$ (notation of Lemma 2), $P$ is collinear with $A$ and $Q'$, and we have $Q'P/AP = -2m/k$. It follows from Lemma 6 with $\lambda = -2m/k$ that $P$ lies on or within the circle (11). Theorem 8 is established.

From a general result on circular regions (§4.2.4, Theorem 4), it follows that under the hypothesis of Theorem 8 all real critical points of $p(z)$ lie in the interval $S$:

$$(- 2mx_1 - k)/(- 2m - k) \leq s \leq (- 2mx_1 + k)/(- 2m - k).$$

In order to study the actual locus of critical points in Theorem 8, we consider a somewhat more general situation:

**Theorem 9.** Let a region $R$ symmetric in the axis of reals be the locus of $q (> 4)$ poles of a real rational function $p(z)$. Then each point of $R$ belongs to the locus of critical points of $p(z)$.

As in the proof of §4.2.2, Theorem 1, we find it convenient to consider $R(z) = 1/p(z)$, and shall prove that each (interior) point of $R$ can be a critical point not a multiple zero of $R(z)$. This conclusion is valid for a real point $\alpha_0$ of $R$, for let $\alpha_0$ lie interior to $R$ on the axis of reals; precisely the method of §4.2.2 then shows that $\alpha_0$ can be a critical point not a multiple zero of $R(z)$.

Let now $\alpha_0$ be a nonreal point of $R$; we set

$$R(z) = (z - \alpha_0)(z - \overline{\alpha_0})(s - \alpha)(s - \overline{\alpha})R_1(z),$$
where $\alpha$ is allowed to vary in the neighborhood of the fixed point $\alpha_0$, but where the zeros and poles of $R_1(z)$ are fixed and remote from $\alpha_0$. The critical points of $R(z)$ are the zeros of $R'(z) = (z - \alpha_0)(z - \alpha) \cdot (z - \bar{\alpha})R_1(z) + (z - \alpha_0)(z - \alpha)(z - \bar{\alpha})R_1(z) + (z - \alpha_0)(z - \alpha)(z - \bar{\alpha})(z - \alpha)R(z) + (z - \alpha_0)(z - \alpha_0)(z - \alpha)(z - \bar{\alpha})(z - \alpha)(z - \bar{\alpha})R'(z) = 0$. This equation defines $z$ as an implicit function of $\alpha$, and the equation is satisfied for $z = \alpha = \alpha_0$. Even though $z$ is not an analytic function of $\alpha$, we have by differentiation for the values $z = \alpha = \alpha_0$

$$\frac{\partial R'(z)}{\partial z} = 2(\alpha_0 - \bar{\alpha}_0)^2 R_1(\alpha_0),$$

$$\frac{\partial R'(z)}{\partial \alpha} = - (\alpha_0 - \bar{\alpha}_0)^2 R_1(\alpha_0).$$

We interpret $R'(z) = 0$ as two real equations in four real variables, the real and pure imaginary parts of $z$ and $\alpha$; the jacobian for the values $z = \alpha = \alpha_0$ has a value which is different from zero. It follows from the implicit function theorem for real variables that the equation $R'(z) = 0$ defines $z$ as a function of $\alpha$, and when $\alpha$ varies throughout a neighborhood of $\alpha_0$ then $z$ also varies throughout a neighborhood of $\alpha_0$; it is readily shown also from the proof of the implicit function theorem based on successive approximations that if $\alpha_0$ is now allowed to vary over a small neighborhood, then a neighborhood of $\alpha_0$ of fixed size as a locus of $z$ which contains uniformly a circle of constant positive radius whose center is the variable $\alpha_0$. The proof of Theorem 9 can now be completed as was the proof of §4.2.2, Theorem 1.

We return now to Theorem 8. If we replace the problem represented by Theorem 8 by the more general problem of finding the critical points of a function of the form $(0 < k < -2m)$

$$p(z) = \frac{(z - x_1)^{k}(z - \alpha_1)^{m_1}(z - \bar{\alpha}_1)^{m_1}(z - \alpha_2)^{m_2}(z - \bar{\alpha}_2)^{m_2}}{\cdots (z - \alpha_n)^{m_n}(z - \bar{\alpha}_n)^{m_n}},$$

where we have $|x_1| < 1$, $|\alpha_j| \leq 1$, $m_j > 0$, $m_1 + m_2 + \cdots + m_n = -m$, and where $k$, $m$, $x_1$ are given but $k$, $m_1$, $m_2$, $\cdots$, $m_n$, $m$ need no longer be integral, then the interior of $C$ plus the closed interior of (11) plus $S$ plus the point at infinity is the precise locus of critical points of $p(z)$. Any point $z$ interior to $C$ can be a critical point of $p(z)$, as follows from Theorem 9. Any point of $S$ can be a critical point of $p(z)$; compare §4.2.4, Theorem 4. Any nonreal point $P$ in the closed interior of (9) not interior to $C$ can be a critical point of $p(z)$, for if $P$ is given there
exists a point $Q'$ on the line $PA$ extended, which is contained in the closed region $R_0$ (notation of Lemma 2) and with $Q'P/AP = -2m/k$; it follows from the proof of Lemma 2 that a suitable choice of negative particles in $C$ of total mass $2m$ is equivalent (so far as concerns the force at $P$) to a $(2m)$-fold negative particle at $Q'$; thus $P$ is a critical point of a suitably chosen $p(z)$ of type (12). Every real point interior to (11) lies in $C$ or $S$, and the conclusion of Theorem 8 persists, so the locus of critical points of (12) is as stated.

Under the original conditions of Theorem 8, with $x_1$ given, every real point interior to $C$ belongs to the locus of critical points if we have $-m = 1$, and every point interior to $C$ belongs to the locus if we have $-m > 1$.

If the hypothesis of Theorem 8 is modified slightly, the conclusion requires large modification:

**Theorem 10.** Let $p(z)$ be a rational function of form (12) where we have $|x_1| < 1$, $|\alpha_j| \leq 1$, $m_j > 0$, $m_1 + m_2 + \cdots + m_n = -m$, and where $m, x_1$ are given, but $k, m_1, m_2, \ldots, m_n, m$ need not be integral. Then the locus of critical points of $p(z)$ for all possible choices of $k, m_1, m_2, \ldots, m_n, \alpha_j$ consists of the extended plane.

The proof of Theorem 10 is similar to the previous proof of the locus property of (12), and is left to the reader.

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