SUMS AND PRODUCTS OF ORDERED SYSTEMS

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1. Introduction. In this paper we state necessary and sufficient conditions for an ordered sum of ordered systems to be a lattice. Other results concerning ordered sums, similar to those given in [3, p. 39] concerning the ordinal power of an ordered system, are obtained. Also a number of analogous results relating to ordered products of ordered systems are given.

We shall use some of the notation and definitions of [3]. For the sake of convenience we list here some of those definitions and symbols that will be employed. By an ordered system is meant a nonempty set $R$ of elements in which a reflexive binary relation $r \geq r'$ is defined. Unless otherwise specified, an italic capital letter always will denote an ordered system in the sequel. A subsystem $T$ of $R$ is a subset of elements of $R$ with the order relation in $T$ imposed by that in $R$.

The expressions and symbols maximal element, greatest element, ascending chain condition, isomorphic, $>$, and so on, will have their usual meanings (see [2], for example). The symbols $\lor$ and $\land$ will be used in denoting least upper bound (l.u.b.) and greatest lower bound (g.l.b.) respectively. The symbols $0$ and $1$ will denote the bounds of bounded ordered systems. The term number will mean partially ordered set. The symbol $S \succ R$ will mean that $R$ is isomorphic to a subsystem of $S$.

If for each element $r$ in $R$, $S_r$ is an ordered system, the ordered sum over $R$ of the systems $S_r$ (denoted by $\sum_{r \in R} S_r$) is the system $P$ where the elements of $P$ are the ordered pairs $(r, s)$ with $r$ in $R$ and $s$ in $S_r$, and $(r, s) \geq (r', s')$ means that $r > r'$ or else $r = r'$ and $s \geq s'$. If all $S_r = S$, we write $R \circ S$ for $\sum_{r \in R} S_r$. The ordered product over $R$ of the $S_r$ (denoted by $\prod_{r \in R} S_r$) is the system $P$ where the elements of $P$ are the functions $f$ defined on $R$ such that $f(r) \in S_r$, while $f \geq f'$ means that if $f(r) \neq f'(r)$, there exists $r' \geq r$ such that $f(r') > f'(r')$.

We list several results in [3] that are used in proofs in this paper.

[3, 2.2] states that $\sum_{r \in R} S_r \succ R$ and $\sum_{r \in R} S_r \succ S_t$ for every element $t$ in $R$.

[3, 2.4] says that $\sum_{r \in R} S_r$ is a number if and only if $R$ and all $S_r$ are numbers.

[3, 3.9] states that $\prod_{r \in R} S_r \succ S_t$ for every element $t$ in $R$.

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1 Numbers in brackets refer to the bibliography at the end of the paper.
Let $R_1$ be the subsystem of $R$ consisting of all $r$ such that $S_r$ is not a cardinal. In part, [3, 3.10] states that if all $S_r$ are numbers, then $\prod R S_r \geq R_1$. [3, 4.13] states that $\prod R S_r$ is a number if and only if (1) all $S_r$ and $R_1$ are numbers, (2) $R_1$ satisfies the ascending chain condition, and (3) if $r'' \geq r' \geq r$ in $R$, if $r''$ and $r' \in R_1$, and $S_r$ contains more than one element, then $r'' \geq r$.

2. Ordered sums.

Theorem 1. $\sum R S_r$ is a lattice [complete lattice] if and only if the following conditions are satisfied.

1. $R$ is a lattice [complete lattice], and all $S_r$ are numbers.

2. Let $P$ be any two element subsystem [any subsystem] of $R$, let $t$ be the l.u.b. of $P$, and let $x$ be the g.l.b. of $P$. If $t \notin P$, then $S_t$ contains 0; if $x \in P$, then $S_x$ contains 1.

3. Let $r$ be any element of $R$, and let $Q$ be any two element subsystem [any subsystem] of $S_r$. If there is no l.u.b. of $Q$, then there is no upper bound of $Q$, $r$ has a least proper successor, say $w$, and $S_w$ contains 0. If there is no g.l.b. of $Q$, then there is no lower bound of $Q$, $r$ has a greatest proper predecessor, say $y$, and $S_y$ contains 1.

Proof. Assume that $\sum R S_r$ is a lattice [complete lattice]. Then it is a number, and [3, 2.4] implies that $R$ and all $S_r$ are numbers. Let $P$ be a two element subsystem [any subsystem] of $R$. Suppose that there is no greatest element of $P$. Let $s_p$, for each element $p$ in $P$, be a fixed element of $S_p$. Then $\vee_{p \in P}(p, s_p) = (\text{l.u.b. of } P, 0)$. By a dual argument it is seen that conditions (1) and (2) are satisfied. Let $r$ be any element of $R$, and let $Q$ be any two element subsystem [any subsystem] of $S_r$. Suppose that there is no l.u.b. of $Q$. Then $\vee_{q \in Q}(r, q) = (w, 0)$, where $w$ must be the least proper successor of $r$. Again a dual argument shows that (3) is satisfied.

Now assume that conditions (1), (2), and (3) hold. (1) and [3, 2.4] together imply that $\sum R S_r$ is a number.

Let $T$ be any two element subsystem [any subsystem] of $\sum R S_r$. Let $P$ be the subsystem of $R$ of all $p$ such that there is an element $(p, s)$ in $T$. If there is a greatest element $u$ of $P$, let $Q$ be the subsystem of $S_u$ of all elements $q$ such that $(u, q) \in T$. Then the l.u.b. of $T$ is $(u, \text{l.u.b. of } Q)$ if $Q$ has a l.u.b. If $Q$ has no l.u.b., then the l.u.b. of $T$ is $(w, 0)$, where $w$ is the least proper successor of $u$.

Suppose that there is no greatest element of $P$. Then the l.u.b. of $T$ is $\text{(l.u.b. of } P, 0)$. A dual argument completes the proof that $\sum R S_r$ is a lattice [complete lattice].

We list several results in the following table.
Property of $\sum_{R} S_r$

Necessary and sufficient conditions on $R$ and the $S_r$

1. Chain
   \( R \) and all $S_r$ are chains.

2. Ordinal
   \( R \) and all $S_r$ are ordinals.

3. Cardinal
   \( R \) and all $S_r$ are cardinals.

4. Bounded number
   All $S_r$ are numbers, $R$ is a bounded number, $S_0$ contains 0, $S_1$ contains 1.

5. Finite number
   \( R \) and all $S_r$ are finite numbers.

The necessity proofs of the above results follow almost immediately from [3, 2.2]. If $\sum_{R} S_r$ is a bounded number, its lower bound is (0, 0), and its upper bound is (1, 1).

The sufficiency proofs are supplied easily with the help of [3, 2.4].

3. Ordered products. In this section the symbol $R_1$ will denote the subsystem of $R$ of all $r$ such that $S_r$ is not a cardinal. References to $R_1$ are to be considered to be deleted when $R_1$ does not exist.

A set of necessary and sufficient conditions that an ordered product, $\prod_{R} S_r$, be a lattice can be given. However, the conditions are so inelegant that the statement of the conditions and the required proof is being omitted from this paper.

**Theorem 2.** $\prod_{R} S_r$ is a complete lattice if and only if the following conditions are satisfied.

1. All $S_r$ are complete lattices.
2. $R_1$ is a number.
3. $R_1$ satisfies the ascending chain condition.

**Proof.** Assume that $\prod_{R} S_r$ is a complete lattice. Then it is a number, and [3, 4.13] implies that $R_1$ and all $S_r$ are numbers and that (3) holds.

Let $r$ be in $R - R_1$. We shall show that $S_r$ is a complete lattice. Suppose that $S_r$ contains two distinct elements, say $a$ and $b$. Let $f_1(r) = a$, $f_2(r) = b$, and $f_1(w) = f_2(w)$ for $w \neq r$ in $R$. Let $f = f_1 \vee f_2$ in $\prod_{R} S_r$.

Now let $k(w) = f(w)$ for $w \neq r$, and let $k(r)$ be an element that does not equal $f(r)$. Then $k$ does not follow or equal $f$, but it can be checked easily that $k$ is an upper bound of $f_1$ and $f_2$. Hence $S_r$ must contain only one element and so be a complete lattice.

Suppose that for some $r$ in $R_1$, $S_r$ is not a complete lattice. By (3), we may assume that if $w > r$ in $R$, $S_w$ is a complete lattice. Suppose that $Q$ is a subsystem of $S_r$ with no l.u.b. Let $f_q(r) = q$ for each $q$ in $Q$. Let $f_q(w) = I$ for any $w > r$ in $R$, and let $f_q(w) = s_w$, where $s_w$ is a fixed element of $S_w$, for each remaining element $w$ in $R$, for each element $q$ in $Q$. The necessity proofs of the above results follow almost immediately from [3, 2.2]. If $\sum_{R} S_r$ is a bounded number, its lower bound is (0, 0), and its upper bound is (1, 1).

The sufficiency proofs are supplied easily with the help of [3, 2.4].
Q. Then it is clear that $\bigvee_{r\in R} \mathcal{S}_r$ does not exist. A dual argument shows that for every $r$ in $R$, $\mathcal{S}_r$ is a complete lattice.

Now assume that conditions (1), (2), and (3) hold. (1) implies that for $r$ in $R - R_1$, $\mathcal{S}_r$ contains just one element. Therefore by [3, 4.13], $\prod_{r\in R} \mathcal{S}_r$ is a number.

The rest of the proof and the sufficiency proof of [3, (7), p. 39] are almost identical.

Let $f_p$, $p$ in $P$, be the elements of any subsystem of $\prod_{r\in R} \mathcal{S}_r$. For $r$ in $R - R_1$, let $f(r) = f_p(r)$. For $r$ a maximal element of $R_1$, let $f(r) = \bigvee_{p\in P} f_p(r)$. Then for any $r$ in $R_1$ such that $f$ is defined over the set $E_r$ of all proper successors of $r$ and $f \geq f_p$ over $E_r$ for all $p$ in $P$, define $P(r)$ to be the set of all $p$ such that $f_p(r') = f(r')$ for all $r' > r$. Let $f(r) = \bigvee_{p\in P} f_p(r)$ if $P(r)$ is not empty; let $f(r) = 0$ if $P(r)$ is empty.

Then $f \geq f_p$ over the system consisting of $r$ and its successors. Since (3) holds, this process defines $f$ on all of $R$ by transfinite induction. As in [3, p. 39], it can be shown that $f = \bigvee_{r\in R} f_r$. A dual proof would show that $\prod_{r\in R} \mathcal{S}_r$ is a complete lattice.

Several other results are listed in the following table.

<table>
<thead>
<tr>
<th>Property of $\prod_{r\in R} \mathcal{S}_r$</th>
<th>Necessary and sufficient conditions on $R$ and the $\mathcal{S}_r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) Chain</td>
<td>$R_1$ and all $\mathcal{S}_r$ are chains, $R_1$ satisfies the ascending chain condition.</td>
</tr>
<tr>
<td>(2) Ordinal</td>
<td>$R_1$ is a finite ordinal, all $\mathcal{S}_r$ are ordinals.</td>
</tr>
<tr>
<td>(3) Cardinal</td>
<td>All $\mathcal{S}_r$ are cardinals.</td>
</tr>
<tr>
<td>(4) Bounded number</td>
<td>All $\mathcal{S}_r$ are bounded numbers, $R_1$ is a number, $R_1$ satisfies the ascending chain condition.</td>
</tr>
<tr>
<td>(5) Finite number</td>
<td>$R_1$ is a number; all $\mathcal{S}_r$ are finite numbers; the set of all $r$ in $R$ such that $\mathcal{S}_r$ contains more than one element is finite; if $r'' \geq r' \geq r$ in $R$, if $r''$ and $r' \in R_1$, and $\mathcal{S}_r$ contains more than one element, then $r''' \geq r$.</td>
</tr>
</tbody>
</table>

The necessity proofs of the above results follow almost immediately from [3, 3.9], [3, 3.10], and [3, 4.13]. If $\prod_{r\in R} \mathcal{S}_r$ is a bounded number and $f = I$, then $f(r) = I$ for all $r$ in $R$; if $f = 0$, then $f(r) = 0$ for all $r$ in $R$.

The sufficiency proofs of these results are supplied easily with the help of [3, 4.13].

By letting $R$ be the number consisting of the integers 1 and 2 ordered by magnitude, we could obtain as corollaries of our results stated so far several easily proved or known results (see, for example, [1] and [2]) concerning the ordinal sum and product of two ordered...
systems. Similarly, results for the cardinal sum and product of ordered systems and some of the results in [3] concerning the ordinal power of an ordered system can be obtained.

We shall state just one corollary of Theorem 1 giving necessary and sufficient conditions for the ordinal product of two ordered systems to be a lattice. Our excuse for stating this corollary is that the conditions as given in [2, p. 25, Ex. 2] are not necessary, as is stated there, but are only sufficient.

**Corollary.** \( R \circ S \) is a lattice if and only if the following conditions are satisfied.

1. \( R \) is a lattice, and \( S \) is a number.
2. If \( R \) is not a chain, then \( S \) is bounded.
3. If there are two elements in \( S \) without a l.u.b., then they have no upper bound, \( S \) contains 0, and every element in \( R \) has a least proper successor.
4. If there are two elements in \( S \) without a g.l.b., then they have no lower bound, \( S \) contains 1, and every element in \( R \) has a greatest proper predecessor.

**Bibliography**


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