SUMS AND PRODUCTS OF ORDERED SYSTEMS

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1. **Introduction.** In this paper we state necessary and sufficient conditions for an ordered sum of ordered systems to be a lattice. Other results concerning ordered sums, similar to those given in [3, p. 39][1] concerning the ordinal power of an ordered system, are obtained. Also a number of analogous results relating to ordered products of ordered systems are given.

We shall use some of the notation and definitions of [3]. For the sake of convenience we list here some of those definitions and symbols that will be employed. By an *ordered system* is meant a nonempty set R of elements in which a reflexive binary relation $r \geqq r'$ is defined. Unless otherwise specified, an italic capital letter always will denote an ordered system in the sequel. A *subsystem* T of R is a subset of elements of R with the order relation in T imposed by that in R.

The expressions and symbols *maximal element, greatest element, ascending chain condition, isomorphic,* $>$, and so on, will have their usual meanings (see [2], for example). The symbols $\vee$ and $\wedge$ will be used in denoting least upper bound (l.u.b.) and greatest lower bound (g.l.b.) respectively. The symbols 0 and I will denote the bounds of bounded ordered systems. The term *number* will mean partially ordered set. The symbol $S > R$ will mean that R is isomorphic to a subsystem of S.

If for each element r in R, S_r is an ordered system, the *ordered sum* over R of the systems S_r (denoted by $\sum_R S_r$) is the system P where the elements of P are the ordered pairs (r, s) with r in R and s in S_r, and $(r, s) \geqq (r', s')$ means that $r > r'$ or else $r = r'$ and $s \geqq s'$. If all $S_r = S$, we write $R \circ S$ for $\sum_R S_r$. The *ordered product* over R of the S_r (denoted by $\prod_R S_r$) is the system P where the elements of P are the functions f defined on R such that $f(r) \in S_r$, while $f \geqq f'$ means that if $f(r) \neq f'(r)$, there exists $r' \geqq r$ such that $f(r') > f'(r')$.

We list several results in [3] that are used in proofs in this paper.

[3, 2.2] states that $\sum_R S_r > R$ and $\sum_R S_r > S_t$ for every element t in R.

[3, 2.4] says that $\sum_R S_r$ is a number if and only if R and all S_r are numbers.

[3, 3.9] states that $\prod_R S_r > S_t$ for every element t in R.

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

Let R_1 be the subsystem of R consisting of all r such that S_r is not a cardinal. In part, [3, 3.10] states that if all S_r are numbers, then $\prod_R S_r > R_1$. [3, 4.13] states that $\prod_R S_r$ is a number if and only if (1) all S_r and R_1 are numbers, (2) R_1 satisfies the ascending chain condition, and (3) if $r'' \geqq r' \geqq r$ in R, if r'' and $r' \in R_1$, and S_r contains more than one element, then $r'' \geqq r$.

2. Ordered sums.

THEOREM 1. *$\sum_R S_r$ is a lattice [complete lattice] if and only if the following conditions are satisfied.*

(1) *R is a lattice [complete lattice], and all S_r are numbers.*

(2) *Let P be any two element subsystem [any subsystem] of R, let t be the* l.u.b. *of P, and let x be the* g.l.b. *of P. If $t \notin P$, then S_t contains* 0; *if $x \notin P$, then S_x contains I.*

(3) *Let r be any element of R, and let Q be any two element subsystem [any subsystem] of S_r. If there is no* l.u.b. *of Q, then there is no upper bound of Q, r has a least proper successor, say w, and S_w contains* 0. *If there is no* g.l.b. *of Q, then there is no lower bound of Q, r has a greatest proper predecessor, say y, and S_y contains I.*

PROOF. Assume that $\sum_R S_r$ is a lattice [complete lattice]. Then it is a number, and [3, 2.4] implies that R and all S_r are numbers. Let P be a two element subsystem [any subsystem] of R. Suppose that there is no greatest element of P. Let s_p, for each element p in P, be a fixed element of S_p. Then $\bigvee_{p \in P}(p, s_p) = (\text{l.u.b. of } P, 0)$. By a dual argument it is seen that conditions (1) and (2) are satisfied. Let r be any element of R, and let Q be any two element subsystem [any subsystem] of S_r. Suppose that there is no l.u.b. of Q. Then $\bigvee_{q \in Q}(r, q) = (w, 0)$, where w must be the least proper successor of r. Again a dual argument shows that (3) is satisfied.

Now assume that conditions (1), (2), and (3) hold. (1) and [3, 2.4] together imply that $\sum_R S_r$ is a number.

Let T be any two element subsystem [any subsystem] of $\sum_R S_r$. Let P be the subsystem of R of all p such that there is an element (p, s) in T. If there is a greatest element u of P, let Q be the subsystem of S_u of all elements q such that $(u, q) \in T$. Then the l.u.b. of T is (u, l.u.b. of Q) if Q has a l.u.b. If Q has no l.u.b., then the l.u.b. of T is $(w, 0)$, where w is the least proper successor of u.

Suppose that there is no greatest element of P. Then the l.u.b. of T is (l.u.b. of P, 0). A dual argument completes the proof that $\sum_R S_r$ is a lattice [complete lattice].

We list several results in the following table.

Property of $\sum_R S_r$	*Necessary and sufficient conditions on* R *and the* S_r
(1) Chain	R and all S_r are chains.
(2) Ordinal	R and all S_r are ordinals.
(3) Cardinal	R and all S_r are cardinals.
(4) Bounded number	All S_r are numbers, R is a bounded number, S_0 contains 0, S_I contains I.
(5) Finite number	R and all S_r are finite numbers.

The necessity proofs of the above results follow almost immediately from [3, 2.2]. If $\sum_R S_r$ is a bounded number, its lower bound is $(0, 0)$, and its upper bound is (I, I).

The sufficiency proofs are supplied easily with the help of [3, 2.4].

3. **Ordered products.** In this section the symbol R_1 will denote the subsystem of R of all r such that S_r is not a cardinal. References to R_1 are to be considered to be deleted when R_1 does not exist.

A set of necessary and sufficient conditions that an ordered product, $\prod_R S_r$, be a lattice can be given. However, the conditions are so inelegant that the statement of the conditions and the required proof is being omitted from this paper.

THEOREM 2. $\prod_R S_r$ *is a complete lattice if and only if the following conditions are satisfied.*

(1) *All* S_r *are complete lattices.*

(2) R_1 *is a number.*

(3) R_1 *satisfies the ascending chain condition.*

PROOF. Assume that $\prod_R S_r$ is a complete lattice. Then it is a number, and [3, 4.13] implies that R_1 and all S_r are numbers and that (3) holds.

Let r be in $R-R_1$. We shall show that S_r is a complete lattice. Suppose that S_r contains two distinct elements, say a and b. Let $f_1(r)=a$, $f_2(r)=b$, and $f_1(w)=f_2(w)$ for $w\neq r$ in R. Let $f=f_1\vee f_2$ in $\prod_R S_r$.

Now let $k(w)=f(w)$ for $w\neq r$, and let $k(r)$ be an element that does not equal $f(r)$. Then k does not follow or equal f, but it can be checked easily that k is an upper bound of f_1 and f_2. Hence S_r must contain only one element and so be a complete lattice.

Suppose that for some r in R_1, S_r is not a complete lattice. By (3), we may assume that if $w>r$ in R, S_w is a complete lattice. Suppose that Q is a subsystem of S_r with no l.u.b. Let $f_q(r)=q$ for each q in Q. Let $f_q(w)=I$ for any $w>r$ in R, and let $f_q(w)=s_w$, where s_w is a fixed element of S_w, for each remaining element w in R, for each element q in

Q. Then it is clear that $\bigvee_{q\in Q} f_q$ does not exist. A dual argument shows that for every r in R, S_r is a complete lattice.

Now assume that conditions (1), (2), and (3) hold. (1) implies that for r in $R-R_1$, S_r contains just one element. Therefore by [3, 4.13], $\prod_R S_r$ is a number.

The rest of the proof and the sufficiency proof of [3, (7), p. 39] are almost identical.

Let f_p, p in P, be the elements of any subsystem of $\prod_R S_r$. For r in $R-R_1$, let $f(r)=f_p(r)$. For r a maximal element of R_1, let $f(r) = \bigvee_{p\in P} f_p(r)$. Then for any r in R_1 such that f is defined over the set E_r of all proper successors of r and $f \geqq f_p$ over E_r for all p in P, define $P(r)$ to be the set of all p such that $f_p(r')=f(r')$ for all $r'>r$. Let $f(r)=\bigvee_{p\in P(r)} f_p(r)$ if $P(r)$ is not empty; let $f(r)=0$ if $P(r)$ is empty. Then $f\geqq f_p$ over the system consisting of r and its successors. Since (3) holds, this process defines f on all of R by transfinite induction. As in [3, p. 39], it can be shown that $f=\bigvee_{p\in P} f_p$. A dual proof would show that $\prod_R S_r$ is a complete lattice.

Several other results are listed in the following table.

Property of $\prod_R S_r$	*Necessary and sufficient conditions on R and the* S_r
(1) Chain	R_1 and all S_r are chains, R_1 satisfies the ascending chain condition.
(2) Ordinal	R_1 is a finite ordinal, all S_r are ordinals.
(3) Cardinal	All S_r are cardinals.
(4) Bounded number	All S_r are bounded numbers, R_1 is a number, R_1 satisfies the ascending chain condition.
(5) Finite number	R_1 is a number; all S_r are finite numbers; the set of all r in R such that S_r contains more than one element is finite; if $r''\geqq r'\geqq r$ in R, if r'' and $r'\in R_1$, and S_r contains more than one element, then $r''\geqq r$.

The necessity proofs of the above results follow almost immediately from [3, 3.9], [3, 3.10], and [3, 4.13]. If $\prod_R S_r$ is a bounded number and $f=I$, then $f(r)=I$ for all r in R; if $f=0$, then $f(r)=0$ for all r in R.

The sufficiency proofs of these results are supplied easily with the help of [3, 4.13].

By letting R be the number consisting of the integers 1 and 2 ordered by magnitude, we could obtain as corollaries of our results stated so far several easily proved or known results (see, for example, [1] and [2]) concerning the ordinal sum and product of two ordered systems. Similarly, results for the cardinal sum and product of ordered systems and some of the results in [3] concerning the ordinal power of an ordered system can be obtained.

We shall state just one corollary of Theorem 1 giving necessary and sufficient conditions for the ordinal product of two ordered systems to be a lattice. Our excuse for stating this corollary is that the conditions as given in [2, p. 25, Ex. 2] are not necessary, as is stated there, but are only sufficient.

COROLLARY. *$R \circ S$ is a lattice if and only if the following conditions are satisfied.*

(1) *R is a lattice, and S is a number.*

(2) *If R is not a chain, then S is bounded.*

(3) *If there are two elements in S without a* l.u.b., *then they have no upper bound, S contains* 0, *and every element in R has a least proper successor.*

(4) *If there are two elements in S without a* g.l.b., *then they have no lower bound, S contains I, and every element in R has a greatest proper predecessor.*

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