ON THE DISTRIBUTION OF THE ROOTS OF A POLYNOMIAL WITH INTEGRAL COEFFICIENTS

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The following question has been raised by D. H. Lehmer in connection with prime number problems: "If \( \epsilon \) is a positive quantity, to find a polynomial of the form \( f(z) = z^n + a_1 z^{n-1} + \cdots + a_n \) where the \( a_i \)'s are integers, such that the absolute value of the product of those roots of \( f \) which lie outside the unit circle, lies between 1 and \( 1 + \epsilon \)." Lehmer calls this absolute value \( \Omega(f) \). Since, for \( f(z) = z^2 - z - 1 \), \( \Omega(f) = 1.6 \cdots \), we may assume \( \epsilon < 1 \), and therefore \( a_n = \pm 1 \). Then \( \Omega(f) \) may be written in the form \( \prod_{n=1}^{r} |z_n|^{||z_n||} \), where the \( z_n \) are the roots of \( f(z) \). It is shown in this paper that no nonreciprocal polynomial \( f(z) \) exists with \( \Omega(f) < 1.179 \). Lehmer himself gives an example of a reciprocal polynomial of degree 10 for which \( \Omega(f) \) is less than this, namely \( 1.176 \cdots \); no nonreciprocal polynomial seems to be known for which \( \Omega(f) < 1.32 \). Thus the present result is still far from being a complete answer to the question.

Let \( f(z) = z^n + a_1 z^{n-1} + \cdots + 1 \) be a nonreciprocal polynomial with integral coefficients, irreducible in the rational field. The proof makes use of the resultant \( R(f, g) \) of \( f(z) \) and the polynomial \( g(z) = \pm z^n f(1/z) = z^n + \cdots \). All the roots of \( g(z) \) are of the form \( z^* = \bar{z}_n \) where the \( \bar{z}_n \) are the conjugates of the roots \( z_n \) of \( f(z) \). Since \( f(z) \) and \( g(z) \) are irreducible and different, have integral coefficients and highest coefficients 1, \( |R(f, g)| = \prod_{n=1}^{r} \prod_{m>n} |z_n - z_m^*| \) must be not less than 1.

\[
| R(f, g) | = \prod_{n=1}^{r} \prod_{m>n} |z_n - z_m^*| \cdot |z_n - z_n^*| \cdot \prod_{n=1}^{r} |z_n - z_n^*| = P_1 \cdot P_2.
\]

If \( \Omega(f) = k \), it will be shown in Lemma 2 that \( P_1 < k^{2r} \cdot r^r \), and in Lemma 3 that \( P_2 \leq (4(k-1)/r)^r \).

**Lemma 1.** \( P_1 = \prod_{n=1}^{r} \prod_{m>n} |z_n - z_m^*| \cdot |z_n - z_n^*| \) has the greatest value compatible with the two conditions (I) \( \prod_{n=1}^{r} |z_n| = 1 \), (II) \( \prod_{n=1}^{r} |z_n| \cdot |z_n||z_n||z_n| = k \) (1 < \( k < 2 \)) for \( |z_1| = k \), \( |z_2| = \cdots = |z_{r-1}| = k \), \( |z_r| = 1/k \).

**Proof.** For \( n = 1, 2, \cdots, r \), call \( b_n \) values of the \( z_n \) for which \( P_1 \) takes its greatest value when restricted by conditions I and II.
Assume that at least two of the \( b_i \)'s are absolutely greater than 1, for example, \( |b_1| > 1, |b_2| > 1 \). \( P_1 \) may be rewritten in the form

\[
P_1 = \prod_{n=3}^{r} (z_1 - z_n^*)(z_2 - z_1^*),
\]

where

\[
\begin{align*}
P_1 &= \prod_{n=3}^{r} \prod_{m>n} (z_n - z_m^*)(z_m - z_n^*) \\
&= \prod_{n=2}^{r-1} |z_1 - z_n|^2 |z_1 - z_n^*|^2 |z_2 - z_n|^2 |z_2 - z_n^*|^2 \\
&= \prod_{n=2}^{r-1} \prod_{m>n} |z_n - z_m|^2 |z_n - z_m^*|^2.
\end{align*}
\]

The expression within the absolute bars is now an analytic function of \( z_1 \) and \( z_2 \) (but not of the remaining \( z_i \)'s). Calling this expression \( F(z_1, z_2, z_3, \ldots, z_r) \), and introducing \( A = b_1 b_2 \) (with \( 1 < |A| \leq k \)), our assumption implies that \( F(z, A/z, b_3, \ldots, b_r) \) which is analytic in \( 1 \leq |z| \leq |A| \) takes its largest value in this region for \( z = z_1 \). But this is impossible, for \( b_1 \) is by assumption an interior point of this region, because \( 1 < |b_1| < |A| \).

Interchanging the \( z_n \) with the \( z_n^* \) does not change \( P_1 \); thus not two of the \( |z_n| \) can be greater than 1, that is, not two of the \( |b_n| \) can be less than 1.

Therefore (by proper arrangement of the subscripts) \( P_1 \) takes its greatest value for \( z_1 = ke^{i\theta_1}, z_1^* = (1/k)e^{i\theta_1}, z_r = (1/k)e^{i\theta_r}, z_r^* = ke^{i\theta_r}, \)

\[
z_n = e^{i\theta_n}, z_n^* = z_n (1 < n < r).
\]

**Lemma 2.** \( P_1 < k^{2r - r^2} \).

**Proof.** Using the values found in Lemma 1, we get, for \( 1 < n < r \)

\[
|z_1 - z_n| \cdot |z_n - z_1^*| = \left| ke^{i\theta_1} - e^{i\theta_n} \right| \cdot \frac{1}{k} \left| ke^{i\theta_n} - e^{i\theta_1} \right|
\]

\[
= \frac{1}{k} \left| z_1 - z_n \right|^2
\]

(since \( \left| ke^{i\theta_n} - e^{i\theta_1} \right| = \left| e^{i\theta_n} - ke^{i\theta_1} \right| \)). Similarly, for \( 1 < n < r \),

\[
\left| z_r - z_n \right| \cdot \left| z_n - z_r^* \right| = k \left| z_r - z_n \right|^2,
\]

\[
\left| z_1 - z_r^* \right| \cdot \left| z_r - z_1^* \right| = \left| e^{i\theta_1} - e^{i\theta_r} \right|^2 < \left| z_1 - z_r \right|^2.
\]

Therefore

\[
P_1 < \left| z_1 - z_r \right|^2 \left| z_1 - z_n \right|^2 \left| z_r - z_n \right|^2 \prod_{n=2}^{r-1} \prod_{m>n} \left| z_n - z_m \right|^2
\]

\[
= \prod_{n=1}^{r} \prod_{m>n} \left| z_n - z_m \right|^2.
\]
The function
\[ G(z) = \frac{1}{z^r} \left( z - \frac{z_1 z_r}{z} \right)^2 \cdot \prod_{n=2}^{r-1} \left( \frac{z_1 z_r}{z} - z_n \right)^2 \cdot \prod_{n=2}^{r-1} \prod_{n<m<r} \left| z_n - z_m \right|^2 \]

is analytic for \( 1 \leq |z| \leq \infty \), and has therefore its greatest absolute value in this region for \( |z| = 1 \). But for \( |z| = 1 \), \( |G(z)| \) is the absolute value of the discriminant of the polynomial whose \( r \) roots \( z, z_1 z_r/z, z_2, \ldots, z_{r-1}, \) all lie on the unit circle. This is known\(^2\) to be not greater than \( r^r \). Therefore \( |G(z)| < r^r \) for \( |z| > 1 \), and \( P_1 < \left| \frac{2^r}{r^r} \right| \cdot |G(z)| \) < \( k^{2r}, r^r \).

**Lemma 3.** If the \( z_n \) satisfy conditions I and II of Lemma 1, then
\[ P_2 = \prod_{n=1}^{s-1} \left| z_n - e_n \right| \leq (4(k-1)/r)^r. \]

**Proof.** If \( \left| z_n \right| = 1 + \epsilon_n \geq 1 \) for \( n = 1, 2, \ldots, s \), and \( \left| z_n^* \right| = \left| z_n \right|^{-1} = 1 + \epsilon_n > 1 \) for \( n = s+1, \ldots, r \), then
\[ \left| z_n - z_n^* \right| = \left| z_n - z_n^{-1} \right| = 1 + \epsilon_n - 1/(1+\epsilon_n) \leq 2\epsilon_n \text{ for } n = 1, 2, \ldots, r. \]
Also, since \( \prod_{n=1}^{s} (1+\epsilon_n) = k \), and \( \prod_{n=s+1}^{r} (1+\epsilon_n) = k, 1 + \sum_{n=1}^{s+1} \epsilon_n \leq k, 1 + \sum_{n=s+1}^{r} \epsilon_n \leq k, \) and \( \sum_{n=1}^{r} \epsilon_n \leq 2(k-1) \). Therefore \( P_2 \leq \prod_{n=1}^{r} (2\epsilon_n) \leq (4(k-1)/r)^r \).

**Theorem.** If \( f(z) \) is a nonreciprocal irreducible polynomial with integral coefficient and highest coefficient 1, then \( \Omega(f) = k > 1.179 \).

**Proof.** If \( r \) is the degree of \( f(z) \), the statement is obviously correct for \( \left| a_r \right| > 1 \), since \( k \geq \left| a_r \right| \). Thus we may assume \( \left| a_r \right| = 1 \). Then by Lemmas 2 and 3, \( |R(f, g)| \equiv P_1 P_2 < k^{2r} r^r (4(k-1)/r)^r = (4k^2(k-1))r \). This must be not less than 1; therefore \( 4k^2(k-1) \geq 1, k > 1.179 \).