Theorems 1 and 2 have their analogues in the case of integrated Lipschitz condition; that is, the following theorem holds:

**Theorem 3.** If \( f(x) \in \text{Lip}(\alpha, p) \), \( 0 < \alpha < 1, \ p \geq 1 \), then for any \( \beta > \alpha \)

\[
\left( \int_0^1 | \sigma_n^{(\beta)}(x; f) - f(x) |^p dx \right)^{1/p} = O(n^{-\alpha}).
\]

Since the proof is analogous to the preceding one, we shall omit it.

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**POSITIVE INFINITIES OF POTENTIALS**

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Let \( R \) denote Euclidean 3-space. The following theorem is due to Evans [1, p. 421].

Let \( E \) be a closed bounded set of capacity zero in \( R \). There exists a distribution of positive mass \( \mu(e) \) entirely on \( E \), such that its potential \( V(M) = \int_R (1/MP) \ d\mu(P) \) is infinite at every point of \( E \) and at no other points.

A proof of the two-dimensional analogue was published by Noshiro [2]. In the present note we show, by a modification of Evans’ construction, that an *absolutely continuous* distribution exists whose potential is infinite on the preassigned set \( E \) only. More precisely, our result, extended to unbounded sets, is as follows:

**Theorem.** Let \( E \) be a closed set of capacity zero in \( R \), and let \( G \) be an open set containing \( E \). Then there exists a non-negative function \( f \) which is summable on \( R \), such that the superharmonic function (that is, the potential)

\[
F(M) = \int_R \frac{1}{MP} f(P) dP
\]

is infinite on \( E \), is continuous in \( R - E \), and is harmonic in \( R - \overline{G} \). (\( \overline{G} \) denotes the closure of \( G \)).

Analogous results evidently hold in two and in more than three dimensions.

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1 Numbers in brackets refer to the references at the end of the paper.

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To begin with, we suppose that $E$ is bounded and contains infinitely many points. Let $n$ be a positive integer, and put

\[ V_n = \max_{q_1, \ldots, q_n \in E} \left\{ \min_{P \in E} \frac{1}{n} \left( \frac{1}{PQ_1} + \cdots + \frac{1}{PQ_n} \right) \right\}. \]

Since $E$ contains infinitely many points, $V_n < +\infty$. The compactness of $E$ implies that there exist points $P_1, \cdots, P_n$ on $E$, such that, for all $P$ on $E$,

\[ \frac{1}{n} \left( \frac{1}{PP_1} + \cdots + \frac{1}{PP_n} \right) \geq V_n. \]

The transfinite diameter of $E$ is defined as the limit of the sequence $\{D_n\}$ where

\[ \frac{n(n-1)}{2} \cdot \frac{1}{D_n} = \min_{q_1, \ldots, q_n \in E} \left\{ \sum_{1 \leq i < j \leq n} \frac{1}{Q_iQ_j} \right\}. \]

It can be shown [1, p. 423] that $V_n \geq (D_{n+1})^{-1}$. The transfinite diameter of a compact set being equal to its capacity [3], it follows that

\[ \lim_{n \to \infty} V_n = +\infty. \]

So far we have followed Evans. We now choose $r_n$ such that $0 < r_n < (nV_n)^{-1}$ (this is possible, since $V_n < +\infty$), and such that the closed spheres $S_i$ with centers at $P_i$ ($i = 1, \cdots, n$) and radius $r_n$ are contained in $G$. For $i = 1, \cdots, n$, we define

\[ \phi_i(P) = \begin{cases} \frac{3}{4nr_n^3} & (P \in S_i), \\ 0 & (P \in R - S_i), \end{cases} \]

\[ u_n(P) = \sum_{i=1}^n \phi_i(P) \quad (P \in R), \]

\[ U_n(M) = \int_R \frac{1}{MP} u_n(P) dP \quad (M \in R). \]

Then $\int_R u_n(P) dP = 1$, and $U_n$ is the potential of a unit mass. Suppose $M \in E$. If $M \in R - (S_1 + \cdots + S_n)$, then

\[ U_n(M) = \frac{1}{n} \left( \frac{1}{MP_1} + \cdots + \frac{1}{MP_n} \right) \geq V_n \]

(by (2)). If $M \in S_i$, then

\[ U_n(M) = \int_{S_i} \frac{1}{MP} \phi_i(P) dP = \frac{3r_n^2 - i^2}{2n^2} \geq \frac{1}{nr_n} > V_n, \]

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where $t = MP_j$. By (4), (5),

$$U_n(M) \geq V_n \quad (M \in E).$$

By (3), there is a sequence $\{n_k\}$ such that $V_{n_k} \geq 2^k$. We define

$$f(P) = \sum_{k=1}^{\infty} 2^{-k}u_{n_k}(P) \quad (P \in R),$$

$$F(M) = \int_R \frac{1}{MP} f(P) dP = \sum_{k=1}^{\infty} 2^{-k}U_{n_k}(M) \quad (M \in R).$$

Then $\int_R f(P) dP = 1$; by (6), $F(M) = +\infty$ if $M \in E$; if $M \in R - E$, $f$ is bounded in some neighborhood of $M$ (since $n \to \infty$ as $n \to \infty$), which implies that $F$ is continuous in $R - E$; and in $R - G$, $f = 0$, hence $F$ is harmonic.

Next, if $E$ is finite, let $E = A_1 + \cdots + A_m$. Choose $r > 0$ such that the closed spheres $S_i$ with centers at $A_i$ ($i = 1, 2, 3, \cdots$) and radius $r$ are contained in $G$, and define

$$\phi(t) = \begin{cases} t^{-2} & \text{if } 0 < t < r, \\ 0 & \text{otherwise}. \end{cases} \quad f(P) = \sum_{i=1}^{m} \phi(PA_i) \quad (P \in R).$$

The conclusion of the theorem evidently holds for this function $f$. Hence the theorem is proved for bounded sets $E$.

Finally, suppose $E$ is unbounded. There exist compact sets $E_i$ ($i = 1, 2, 3, \cdots$) such that $R = \bigcup_{i=1}^{\infty} E_i$, and open sets $G_i$ containing $E_i$ such that no point of $R$ is in more than four of the sets $G_i$. We now apply the previously obtained result for bounded sets to construct functions $f_i$ ($i = 1, 2, 3, \cdots$) which satisfy the conclusion of the theorem with respect to the sets $E_i$, $G_i$, and put $f(P) = \sum_{i=1}^{\infty} f_i(P)$. For any $P$, this sum contains at most four nonzero terms. Hence it is easily verified that the conclusion of the theorem holds.

References