GREEN'S SECOND IDENTITY FOR GENERALIZED LAPLACIANS

WALTER RUDIN

Let $J(P, r)$ denote the closed circular disc bounded by the circle $C(P, r)$ with center at $P$, and radius $r$, in the plane. We define the generalized Laplacian of the function $F$ at $P$ by

$$\Lambda F(P) = \lim_{r \to 0} 4\left[\frac{m(F; P, r) - F(P)}{r^2}\right],$$

where $m(F; P, r)$ is the mean value of $F$ on $C(P, r)$. The upper and lower Laplacians $\Lambda^+ F(P)$ and $\Lambda^- F(P)$ are defined likewise, with $\lim \sup$ and $\lim \inf$ in place of $\lim$ [3].\(^1\)

If $f \in L$ in a bounded domain $R$, we define [3, p. 281]

$$\Omega_R f(P) = -\frac{1}{2\pi} \int \int_R f(Q) g(P, Q) dQ \quad (P \text{ in } R),$$

where $g(P, Q)$ is Green's function for $R$. In [3] we established the existence of $\Lambda F(P)$ for almost all $P$ of a domain $D$, the integrability of $\Lambda F(P)$ over any compact subset of $D$, and the formula

$$F(P) = \Omega_R \Lambda F(P) + H(P),$$

valid for almost all $P$ of any bounded domain $R$ such that $\bar{R} \subset D$, where $H$ is harmonic in $R$ and assumes the values of $F$ on the boundary of $R$, under the following hypotheses:

- (A) $F$ is continuous in $D$;
- (B) $\Lambda^+ F(P) > -\infty$, $\Lambda^- F(P) < +\infty$, except possibly on a closed set of capacity zero;
- (C) there exists a function $\gamma$, defined in $D$, such that $\gamma \in L$ on every compact subset of $D$, and such that $\gamma(P) \leq \Lambda^+ F(P)$ for $P$ in $D$.

In [4], (B) was slightly weakened. In the present paper the above result is used to obtain the following theorem.

**Theorem.** If the functions $U$ and $V$ satisfy (A), (B), (C) in a domain $D$, and if $U(P) = 0$ outside a compact subset $K$ of $D$, then

$$\int \int_D U(P) \Lambda V(P) dP = \int \int_D V(P) \Lambda U(P) dP.$$

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\(^1\) Numbers in brackets refer to the references at the end of the paper.
If $U$ and $V$ have continuous second derivatives, then (2) is clearly an immediate consequence of Green's second identity [1, p. 215].

Let $R$ be a bounded domain such that $K \subseteq R \subseteq D$. Since $U$, and therefore $\Delta U$, vanish outside $K$, it suffices to prove that (2) holds with $R$ in place of $D$.

Putting $u(P) = \Delta U(P)$, $v(P) = \Delta V(P)$, wherever the Laplacians exist, we have, by (1), writing $\Omega$ for $\Omega_R$,

$$U(P) = \Omega u(P), \quad V(P) = \Omega v(P) + H(P) \quad \text{(p.p. in $R$)}.$$ 

By Fubini's theorem, and (3),

$$\int \int_R v(P)\Omega u(P) \, dP = \int \int_R u(P)\Omega v(P) \, dP = \int \int_R u(P)V(P) \, dP - \int \int_R u(P)H(P) \, dP.$$ 

Hence it is enough to prove that

$$\int \int_R u(P)H(P) \, dP = 0$$

for every function $H$ harmonic in $R$. Choose a domain $G$ such that $K \subseteq G \subseteq \overline{G} \subseteq R$. Choose $r > 0$ such that $J(P, 3r) \subseteq R$ if $P \in G$. Define $H(P) = 0$ outside $R$. Put $H_1(P) = A_r H(P)$ (that is, the mean of $H$ on $J(P, r)$), $H_2(P) = A_r H_1(P)$, and $H_3(P) = A_r H_2(P)$, for all $P$. Then $H_3(P) = H(P)$ in $G$, $H_3$ has continuous second derivatives everywhere [2, p. 343], and $H_3(P) = 0$ outside some bounded domain $T$ containing $\overline{G}$. Hence we have, for all $P$ in $T$,

$$H_3(P) = \Omega_T h_3(P),$$

where $h_3(P) = \Delta H_3(P)$. Noting that $u(P) = u(P) = 0$ whenever $H(P) \neq H_3(P)$, we obtain

$$\int \int_T u(P)H(P) \, dP = \int \int_T u(P)\Omega_T h_3(P) \, dP = \int \int_T h_3(P)\Omega_T u(P) \, dP = \int \int_T U(P)\Delta H(P) \, dP = 0,$$

since $\Delta H(P) = 0$ in $K$, and $U(P) = 0$ in $T - K$. This proves (4), and hence the theorem.
The extension to more than two dimensions is evident.

REFERENCES