CERTAIN HOMOGENEOUS UNICOHERENT INDECOMPOSABLE CONTINUA

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A simple closed curve is the simplest example of a compact, non-degenerate, homogeneous plane continuum. If a bounded, nondegenerate, homogeneous plane continuum has any local connectedness property, even of the weakest sort, it is known to be a simple closed curve \([1, 2, 3]\). The recent discovery of a bounded, nondegenerate, homogeneous plane continuum which does \textit{not} separate the plane \([4, 5]\) has given substance to the old question as to whether or not such a continuum must be indecomposable. Under certain conditions such a continuum must \textit{contain} an indecomposable continuum \([6]\). It is the main purpose of this paper to show that every bounded homogeneous plane continuum which does not separate the plane is indecomposable.

**NOTATION.** If \(M\) is a continuum and \(x\) is a point of \(M\), \(U_x\) will be used to denote the set of all points \(z\) of \(M\) such that \(M\) is aposyndetic at \(z\) with respect to \(x\).\(^\text{1}\) It is evident that \(U_x\) is an open subset of \(M\).

**Lemma.** \textit{If the compact metric continuum} \(M\) \textit{is homogeneous and} \(x\) \textit{and} \(y\) \textit{are distinct points of} \(M\), \textit{then} \(U_y\) \textit{is not a proper subset of} \(U_x\).

**Proof.** Suppose on the contrary that \(U_y\) is a proper subset of \(U_x\). Since \(M\) is homogeneous, there exists a homeomorphism \(T\) such that \(T(M) = M\) and \(T(x) = y\). Then \(T(U_x) = U_y\) and \(T(U_y)\) is a proper subset of \(U_x\). Hence there exists a sequence \(x_0 = x\), \(x_1 = y\), \(x_2 = T(y)\), \(\cdots\), \(x_n = T^n(x)\), \(\cdots\) of points of \(M\) such that for each positive integer \(n\), \(U_{x_n}\) is a proper subset of \(U_{x_{n-1}}\). For no two non-negative integers \(i\) and \(j\) is \(x_i = x_j\), because if \(x_i = x_j\) then \(U_{x_i} = U_{x_j}\). Consequently the sequence \(x_1, x_2, x_3, \cdots\) has a limit point \(x_0\). Now for each positive integer \(n\), \(U_{x_n}\) is a subset of \(U_{x_{n-1}}\), because if \(p\) is a point of \(U_{x_n}\) there exist a subcontinuum \(K\) of \(M\) and an open subset \(V\) of \(M\) such that \(M - x_n \supseteq K \cup V \cup p\); hence for infinitely many positive integers \(n\), \(M - x_n \supseteq K \cup V \cup p\); so for infinitely many positive integers \(n\), \(M\) is aposyndetic at \(p\) with respect to \(x_n\) and hence \(p\) belongs to \(U_{x_n}\).

Evidently \(x_n \neq x_n\), \(n = 1, 2, 3, \cdots\). And since \(M\) is homogeneous,

\(855\)

\(^\text{1}\) Numbers in brackets refer to the bibliography at the end of this paper.

\(^\text{2}\) The continuum \(M\) is \textit{aposyndetic} at the point \(z\) of \(M\) with respect to the point \(x\) of \(M\) provided that \(M\) contains a continuum \(K\) and an open (rel. \(M\)) subset \(V\) such that \(M - x \supseteq K \cup V \cup z\).
there exists a homeomorphism $T_1$ such that $T_1(M) = M$ and $T_1(x) = x_\omega$. Then $T_1TT_1^{-1}$ is a homeomorphism of $M$ onto itself such that if we let $x_{\omega+1} = T_1TT_1^{-1}(x_\omega)$, $T_1TT_1^{-1}(U_{x_\omega}) = U_{x_{\omega+1}}$, which is a proper subset of $U_{x_\omega}$. This process can be continued uncountably many times to produce a well-ordered sequence $\alpha = x_1, x_2, x_3, \ldots, x_i, \ldots (i < \omega_1)$, of distinct points of $M$ such that (1) if $x_i$ of $\alpha$ has no immediate predecessor in $\alpha$, $x_i$ is a limit point of some countable subsequence of $\alpha$ running through the terms of $\alpha$ preceding $x_i$ in $\alpha$, and (2) $U_{x_1}, U_{x_2}, U_{x_3}, \ldots, U_{x_i}, \ldots$ is a monotone descending sequence of distinct open subsets of $M$. In a compact metric space (2) is impossible.

**Theorem 1.** A homogeneous, hereditarily unicoherent, compact metric continuum $M$ is indecomposable.

**Proof.** Suppose that $U$ is an open subset of $M$ and $H$ is a subset of $M - U$ such that in order for a point $x$ to belong to $H$ it is necessary and sufficient that $U_x = U$. In case $M$ contains no such sets $U$ and $H$, $M$ is indecomposable by Theorem 9 of [7].

It is rather easy to see that $H$ is closed. Suppose that there exists a point $y$ of $H - H$. Let $z$ be a point of $U_y$. Then $M$ is aposyndetic at $z$ with respect to $y$ and hence $M$ is aposyndetic with respect to some point of $H$. Consequently $z$ belongs to $U$. But by the lemma $U_y$ cannot be a proper subset of $U$ and hence $U_y = U$ and $y$ belongs to $H$. So $H$ is closed.

If $w$ is a point of $M$ such that some point $x$ of $H$ cuts $w$ from a point $z_1$ of $U$, then $x$ cuts $w$ from all points of $U$ and $w$ belongs to $H$. For suppose that $z_2$ is a point of $U$. There exist continua $K_1$ and $K_2$ and open sets $V_1$ and $V_2$ such that $M - x \supset K_1 \supset V_1 \supset z_1$ ($i = 1, 2$). Now if $x$ cuts $V_1$ from $V_2$, it follows from the homogeneity of $M$ that every point of $M$ cuts between two open subsets of $M$; but by Corollary 2 of [8], this is impossible. So $x$ does not cut $V_1$ from $V_2$ and hence there exists a continuum $K$ in $M - x$ such that $K \cdot V_1 \neq 0$ and $K \cdot V_2 \neq 0$. The continuum $K_1 + K + K_2$ contains $z_1$ but not $x$; hence $K_1 + K + K_2$ does not contain $w$; consequently $x$ cuts $w$ from all points $z_2$ of $U$ and furthermore $z_2$ belongs to $U_w$. This shows that $U$ is a subset of $U_w$ and, by the lemma, $U = U_w$. So $w$ belongs to $H$.

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* Throughout this proof $M$ will be considered to be space. If there do not exist points $x$ and $z$ of $M$ such that $M$ is aposyndetic at $z$ with respect to $x$, then $M$ is indecomposable (Theorem 9 of [7]). So because $M$ is homogeneous it will be assumed that for each point $x$ of $M$, $U_x$ exists (that is, nonvacuous).

* A point $x$ cuts $w$ from $z$ (in $M$) provided that there exists no subcontinuum of $M$ lying in $M - x$ and containing $w + z$. 

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For each point $o$ of $H$, let $N_o$ denote $o$ together with all points $x$ of $H$ such that $x$ cuts $o$ from $U$. The set $N_o$ is closed. Now suppose that for some point $o$ of $H$, $o$ does not cut all other points of $N_o$ from $U$. Then $N_o$ contains a point $o_1$ such that $N_{o_1}$ is a subset of $N_o - o$. A homeomorphism of $M$ onto itself carrying $o$ into $o_1$ leaves $U$ invariant and carries $o_1$ into a point $o_2$ of $H$ such that $N_{o_2}$ is a proper subset of $N_{o_1}$. As in the proof of the lemma, this process may be continued uncountably many times to produce an uncountable monotone sequence of distinct closed sets. This is impossible. Consequently $o$ cuts all other points of $N_o$ from $U$. It follows at once that each point of $N_o$ cuts all other points of $N_o$ from $U$ and in particular if a point $p$ of $H$ cuts a point $o$ of $H$ from $U$, then $o$ cuts $p$ from $U$, and $N_o = N_p$.

The set $H$ contains no domain. For suppose on the contrary that $H$ contains a domain $D$. Let $o$ denote a point of $H$. Then $N_o$ does not contain $D$, for if it did, a point $x$ of $D$ could cut the domain $D - x$ from the domain $U$ contrary to Corollary 2 of [8]. So $D - D \cdot N_o$ is a domain in $H$ containing no point of $N_o$. Now $M$ is not aposyndetic at any point of $D - D \cdot N_o$ with respect to a point of $N_o$. Hence by Theorem 6 of [7], if $z$ is a point of $U$, $D - D \cdot N_o$ contains a point $x$ and $N_o$ contains a point $y$ such that $y$ cuts $x$ from $z$ and hence from $U$. Therefore $y$ cuts $x$ from $U$ and consequently $x$ belongs to $N_o$. This is a contradiction since $x$ belongs to $D - D \cdot N_o$. So $H$ contains no domain.

The domain $U$ is dense in $M$. Suppose the contrary. There exists a domain $D$ lying in $M - (U + H)$. Let $y$ be a point of $H$. By the definition of $U$, $M$ is not aposyndetic at any point of $D$ with respect to $y$. Let $z$ be a point of $U$. By Theorem 6 of [7], $D$ contains a point $x$ such that $y$ cuts $x$ from $z$. Hence (by paragraph 3 of this proof) $x$ belongs to $H$ contrary to construction. So $U$ is dense in $M$ and the boundary of $U$ is $M - U$.

The set $M - U$ is a continuum. Obviously $M - U$ is closed. Suppose that $M - U$ is not connected; then $M - U = A + B$ where $A = A$, $B = B$, and $A \cdot B = 0$. Suppose that $A$ contains a point $x$ of $H$. There exists a domain $D$ such that $D$ contains $B$ but $D \cdot A = 0$. Each point of the boundary $\beta$ of $D$ belongs to $U$; so there exist a finite collection $K_1$, $K_2$, $\ldots$, $K_n$ of continua and a collection $V_1$, $V_2$, $\ldots$, $V_n$ of domains such that $V_1$, $V_2$, $\ldots$, $V_n$ covers $\beta$ and for each $i$, $1 \leq i \leq n$, $M - x \supseteq K_i \supseteq V_i$. Since by Corollary 2 of [8] (and the homogeneity of $M$) $x$ does not cut any two domains from each other, there exists a
continuum $K$ in $M - x$ which contains $D$. Hence $M$ is aposyndetic at each point of $B$ with respect to $x$. This is contrary to the definitions of $B$ and $U$. Hence $M - U$ is connected.

Let $o$ be a point of $H$. Then $N_o = M - U$. Suppose on the contrary that $q$ is a point of $M - U$ not in $N_o$. If $q$ cuts $o$ from a point of $U_q$, then $q$ cuts $U_q$ from $o$. Let $T$ be a homeomorphism of $M$ onto itself carrying $o$ into $q$.\* Evidently $T(U) = U_q$ and (by paragraph 3, $T(H)$ taking the role of $H$) $o$ belongs to $T(H)$. Therefore $o$ cuts $q$ from $U_q$.

But $U_q \cap U \neq 0$ since both $U$ and $U_q$ are open, dense subsets of $M$; so $o$ cuts $q$ from a point of $U$. Hence $q$ belongs to $H$. It follows that $q$ cuts $o$ from $U$ and thus belongs to $N_o$. From this contradiction it is evident that no point $q$ of $M - (U + N_o)$ cuts $o$ from a point of $U_q$.

Now let $K$ be a continuum containing a domain $V$ of $U$ and lying in $M - o$. Since each point of $N_o$ cuts every other point of $N_o$ from $U$, $K$ contains no point of $N_o$. Since no point of $N_o$ cuts a point $q$ of $M - (U + N_o)$ from a point of $U$, $K$ may be assumed to contain a point of $M - (U + N_o)$. For each point $q$ of $K$ there exists a continuum $C_q$ from $V$ to $o$ lying in $M - q (V \cdot U_q \neq 0)$. Let $F$ denote a finite collection of these continua, $C_q$, such that if $p$ is a point of $K \cdot [M - (U + N_o)]$ there exists a continuum $C_p$ from $V$ to $o$ lying in $M - p$.

Let $G$ be a collection consisting of $H$ together with every image of $H$ under homeomorphisms of $M$ onto itself. It is easy to see that $G$ fills up $M$ and no two elements of $G$ have a point in common. Furthermore, $G$ is upper-semicontinuous for if some sequence $x_1, x_2, x_3, \cdots$ of points of distinct elements of $G$ converged to a point $x$ of an element of $G$, say $H$, but some infinite sequence $y_1, y_2, y_3, \cdots$ of points from the same elements of $G$ converged to a point $y$ of $M - H$, then $M$ would be aposyndetic at $y$ with respect to $x$. But for each $i$, $M$ is not

\* Roughly stated the purpose of $T$ is merely to shift the frame of reference from $o$ to $q$, so that results already obtained for $H$ and $U$ will apply to similar sets constructed for $q$. 
aposyndetic at $y$, with respect to $x$, and this contradicts Theorem 1 of [7]. So $G$ is upper-semicontinuous.

With respect to its elements as points, $G$ is a continuum $M'$. Furthermore $M'$ is homogeneous and aposyndetic. In such a continuum the meaning of "cut point" and "separating point" are the same [9]. Since $M'$ contains a nonseparating point, every point of $M'$ is a nonseparating point because of the homogeneity. Let $A$ and $B$ denote distinct points of $M'$ and let $T$ denote a continuum in $M'$ irreducible from $A$ to $B$. Let $X$ denote a point of $T-(A+B)$. There exists in $M'-X$ a continuum $T_1$ containing $A+B$. But $M'$ is hereditarily unicoherent. So $T-T_1$ is a subcontinuum of $T$ containing $A+B$ but not $X$. This is a contradiction and from this contradiction Theorem 1 follows.

**Theorem 2.** If $M$ is a homogeneous, bounded, plane continuum which does not separate the plane, $M$ is indecomposable.

Theorem 2 follows immediately from Theorem 1.

The following question remains unanswered: Is every homogeneous, bounded, nondegenerate, plane continuum which does not separate the plane a pseudo-arc?

**Bibliography**


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