the resulting 2-cell be amalgamated with $\alpha_2$ across $c_2$ to obtain a 2-cell $\rho$ with $\rho$ for boundary. By Theorem 2.1, $\rho$ is the interior of $\rho$. Since $\alpha_1$ and $\alpha_2$ are exterior to $c$, the 2-cell $\alpha$ constitutes the entire interior of $c$. The Jordan-Schoenflies Theorem now follows readily in all its generality.

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A NOTE ON CURVATURE AND BETTI NUMBERS

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1. S. Bochner has proved the following theorem [2]: Let $M^{(m)}$ be a closed manifold with complex structure [4; 7] of complex dimension $m$, on which there exists a Kähler-metric [2; 3; 5]

(1) $ds^2 = g_{ik}^* (dz^i dz^k)^3$

(2) $\frac{\partial g_{ik}^*}{\partial z_1} = \frac{\partial g_{ik}^*}{\partial z_i}$

Let $R_{ik}^*$ denote the Ricci tensor and

(3) $P_{hi^* jk^*} = R_{hi^* jk^*} - \frac{1}{m + 1} (g_{hi^*} R_{jk^*} + g_{hk^*} R_{i^* j})$

the tensor of projective curvature. In every point of $M^{(m)}$ we form the numbers

(4) $L = \inf_{i^*} \frac{-R_{hi^*} \xi^i \xi^j}{\xi^i \xi_j}$

(5) $P = \sup_{i^*} \left| \frac{P_{hi^* jk^*} \xi^h \xi^i}{\xi^h \xi^i} \right|$

with all vectors $\xi^i$ and skew-symmetric tensors $\xi^i j^k$ attached to the point in question. If

(6) $L > 0$

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1 Numbers in brackets refer to the bibliography at the end of the paper.
2 Products of differentials in parentheses denote ordinary products, products without parentheses are skew products.
3 We denote by $i^*$ the index relative to $z^i$. 
holds everywhere on $M^{(m)}$, and if, for some $p$, we have at all points the relation

$$\left(\frac{p-1}{m+1}\right)L < 1,$$

then for the Betti numbers $\beta^q$ of $M^{(m)}$ we have

$$\beta^{2k} = 1, \quad 2k \leq p,$$

$$\beta^{2k+1} = 0, \quad 2k + 1 \leq p.$$

2. The proof of this theorem in [2] is based on Bochner’s lemma [1]:

If on a compact (differentiable) manifold we have, for a given scalar $\Psi$, $\Delta \Psi = 0$ everywhere, then $\Delta \Psi = 0$ everywhere.

By the theory of harmonic integrals [5, §2] the group of complex harmonic differential forms of degree $q$ on $M^{(m)}$ is isomorphic to the $q$th cohomology group with real coefficients of $M^{(m)}$. The form

$$\Omega = g_{ik}dz^i \wedge \overline{dz^k}$$

with all its powers $\Omega^k$, $k \leq m$, is harmonic and not equal to $0$ [3; 5; 6]. It is known that every harmonic differential form $\phi^q$ of degree $q \leq m$ is a sum with constant coefficients $c_i$,

$$\phi^q = \sum_{i=0}^{(q/2)} c_i \chi^{q-2i} \Omega^i,$$

where the forms $\chi$ are harmonic and “effective,” that is, satisfy the condition

$$\ast \chi = 0.$$

A differential form $\phi_{(k)}^q$ is said to be pure, and of type $k$, if it is a homogeneous form of degree $k$ in the differentials $dz^i$. Every harmonic form is a sum of pure harmonic forms. Let $\beta_{(k)}^q$ be the rank of the linear space of the pure harmonic forms on $M^{(m)}$ of degree $q$ and type $k$. Then

$$\beta^{2k} = \beta_{(k)}^{2k} \pmod{2},$$

$$\beta^{2k+1} = 0 \pmod{2}.$$ 

(14) has been proved first by Lefschetz in the case of algebraic manifolds.

If $\phi^q$ is a pure form $\phi_{(k)}^q$, then (11) becomes [5]
A NOTE ON CURVATURE AND BETTI NUMBERS

3. We now prove the following theorem.

Theorem. If on $M^{(m)}$ we have (6) and (7) everywhere, then there exist no effective harmonic forms of type $k$ and degree $2k$ for $k \leq p$, that is, we have

$$\phi^{2k} = 1 \pmod{2}, \quad 2k \leq 2p.$$ 

From (8), (9), (14), and (15) we see that we have the following situation if (6) and (7) hold everywhere on $M^{(m)}$:

$$\begin{align*}
\phi^{2k} &= 1 \text{ for } 2k \leq p, \quad \phi^{2k} = 1 \pmod{2} \text{ for } 2k \leq 2p, \\
\phi^{2k+1} &= 0 \text{ for } 2k + 1 \leq p, \quad \phi^{2k} = 0 \pmod{2} \text{ always.}
\end{align*}$$

Proof. It is a fundamental property of a Kähler metric [5] that only those components of the curvature tensor which are equal to one of the form $R_{ij}^{,kl}$ are not equal to 0. Let $\psi_{(k)} = P_{i_1 \cdots i_{k-1} k}^r_{k+1 \cdots i_q}$ be an effective form of degree $q$ and type $k$. Then (12) is

$$g^{r} s^{*} P_{i_1 \cdots i_{k-1} k}^{r} \cdot s^{*} = 0.$$ 

Let $\Phi(\psi_{(k)})$ be the scalar $P_{i_1 \cdots i_{k-1} k}^r_{k+1 \cdots i_q}$ and $k \leq q - k$. Condition (16) gives [2, (36)]

$$\Delta \Phi(\psi_{(k)}) = \frac{1}{2} \Delta \Phi = P_{i_1 \cdots i_{k-1} k}^r_{k+1 \cdots i_q} \cdot s^{*} + \frac{1}{k} \left( (k - 1) P_{i_1 \cdots i_q k}^r_{i_{k-1} k+1 \cdots i_q} - \frac{k - 1}{m + 1} P_{i_1 \cdots i_q k}^r_{i_{k-1} k+1 \cdots i_q} \cdot s^{*} \right).$$

This formula shows that from (6) and (7) follows

$$\Delta \Phi(\psi_{(k)}) > 0, \quad k \leq p,$$ 

a contradiction to Bochner's lemma.

4. The theory of effective harmonic forms and (11) is not restricted to Kähler manifolds [5]. In fact, if on a Riemannian manifold $M^{(m)}$ of dimension $2m$ a 2-form exists,

$$\Omega = h_{ik} dx^i dx^k,$$

which is closed and is everywhere of the same rank $2p$, then (11) is
valid for $q \leq p$, the effective forms being defined analogously to (12) as forms $\phi^q = F_{i_1 \ldots i_q} dx^{i_1} \cdots dx^{i_q}$ satisfying

$$h_{ik} F_{i_1 i_2 \ldots i_q} = 0.$$ 

Let $P_{hijk}$ be defined as

$$P_{hijk} = R_{hijk} - \frac{1}{m+1} (g_{hi} R_{ijk} + h_{hk} R_{ij}).$$

It is easy to see that (6) and (7) imply (8) and (9) for $2k$ resp. $2k+1 \leq \min (p, \rho)$.

**Bibliography**


*BASEL, SWITZERLAND*