A NONCOMMUTATIVE ORDINALLY SIMPLE LINEARLY ORDERED GROUP

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B. H. Neumann[1] has recently given an example of a noncommutative ordinally simple linearly ordered group.[2] This confirmed a conjecture of Garrett Birkhoff[3] that there must exist ordinally simple ordered groups which are not merely subgroups of the additive group of real numbers. In the present note we give a more elementary example.

Let G be the group generated by a set of symbols $g(\alpha)$, one to each rational number α, subject to the generating relations

$$g(\alpha)g(\beta) = g\left(\frac{\alpha+\beta}{2}\right)g(\alpha) \qquad \text{if } \alpha > \beta. \tag{1}$$

Observe that the term of higher index is the same on both sides. By repeated application of (1) it is clear that every element a of G can be brought to "normal form":

$$a = g(\alpha_1)^{m_1}g(\alpha_2)^{m_2} \cdots g(\alpha_r)^{m_r} \tag{2}$$

where the m_i are nonzero integers, and the α_i are rational numbers satisfying $\alpha_1 < \alpha_2 < \cdots < \alpha_r$.

It can be shown that the normal form (2) is unique. In particular, $g(\alpha) \neq g(\beta)$ if $\alpha \neq \beta$. The scheme is to show that if C is a chain of elementary transformations carrying a word W_1 into a word W_2, and if C involves a generator of index higher than any in either W_1 or W_2, then there exists a chain shorter than C carrying W_1 into W_2. We omit the details, but comment that the proof thereof depends upon the fact that $\alpha \circ \beta = (\alpha+\beta)/2$ satisfies the condition[4]

$$\alpha \circ (\beta \circ \gamma) = (\alpha \circ \beta) \circ (\alpha \circ \gamma). \tag{3}$$

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[1] B. H. Neumann, *On ordered groups*, Amer. J. Math. vol. 71 (1949) pp. 1–18; the example is on p. 16.

[2] A linearly ordered group G is said to be ordinally simple if it contains no nontrivial convex normal subgroup. A subset H of G is convex if, with any two elements of G, it contains every element of G lying between them.

[3] Garrett Birkhoff, *Lattice ordered groups*, Ann. of Math. vol. 43 (1942) pp. 298–331; ref. Problem 5, p. 329.

[4] We might call a binary operation "self-distributive" if it satisfies (3). Any group is a self-distributive system under the operation $\alpha \circ \beta = \alpha\beta\alpha^{-1}$, and any ring likewise under $\alpha \circ \beta = (\alpha-\beta)\lambda+\beta$ with fixed λ.

We order G by declaring $a>1$ if $m_r>0$. To see that G is ordinally simple, let H be a convex normal subgroup of G containing an element $b\neq 1$ of G. We can assume that $b>1$. Let $g(\beta)$ be the generator of highest index occurring in b. Since $1<g(\beta)<b^2$, and H is convex, we conclude that $g(\beta)$ is in H. Let γ be any rational number, and let $\alpha=2\gamma-\beta$. If $\gamma<\beta$, then $1<g(\gamma)<g(\beta)$ and $g(\gamma)\in H$. If $\gamma>\beta$, then $\alpha>\beta$ and H, being normal, must contain

$$g(\alpha)g(\beta)g(\alpha)^{-1} = g\left(\frac{\alpha+\beta}{2}\right) = g(\gamma).$$

Thus H contains every generator $g(\gamma)$ of G, and so must coincide with G.

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