PATTERN INTEGRATION WITH IMPROPER RIEMANN INTEGRALS

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1. **Introduction.** In a previous paper [1][1] we have recently introduced and discussed a type of integration, which we have called pattern integration, and have established the following theorem:

THEOREM 1.1. *Let $f(x)$ be (proper) Riemann integrable, $0 \leqq x \leqq 1$. Let $(k-1)/n \leqq \xi_k^{(n)} \leqq k/n$ $(k=1, 2, 3, \cdots, n)$. Let the pattern P be characterized by the dyadic number*

$$t = 0.\alpha_1\alpha_2\alpha_3 \cdots \alpha_n \cdots (2),$$

such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \alpha_k = \alpha.$$

Then the pattern integral

$$(P) \int_0^1 f(x)dx \equiv \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \alpha_k f(\xi_k^{(n)}) = \alpha(R) \int_0^1 f(x)dx.$$

It seems desirable to extend this result, if possible, to improper Riemann integrals. In this paper we shall confine our remarks to improper integrals of the form

$$\oint_0^1 f(x)dx \equiv \lim_{\epsilon\to 0+} (R) \int_\epsilon^1 f(x)dx.$$

(R) in front of any integral will indicate a proper Riemann integral.

2. **The improper integral defined as the limit of a sum.** We first consider the case where $\alpha_k = 1$ $(k=1, 2, 3, \cdots)$. A fundamental difficulty not encountered in the previous paper is apparent from the definition of $\oint_0^1 f(x)dx$. Bromwich and Hardy have considered this problem [2]. As they point out, the ordinary definition of this improper integral is by means of a double (repeated) limit; and in order to replace this limit by the single limit,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(\xi_k^{(n)}),$$

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

some restrictions are certainly necessary. In the first place, if $0<\xi_1^{(n)} \leqq 1/n$, we can clearly choose $\xi_1^{(n)}$ so that $|(1/n)f(\xi_1^{(n)})|$ is greater than any assigned number. It is then suggested that the most obvious choice is $\xi_k^{(n)}=k/n$, and with this restriction they establish:

THEOREM 2.1. *Let $f(x)$ be positive and tend steadily to ∞ as x tends to zero. Let $\oint_0^1 f(x)dx$ be convergent. Then*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \oint_0^1 f(x)dx.$$

The proof rests on the fact that for all n,

$$\frac{1}{n}\sum_{k=2}^{n} f\left(\frac{k}{n}\right) \leqq (R)\int_{1/n}^{1} f(x)dx \leqq \frac{1}{n}\sum_{k=1}^{n-1} f\left(\frac{k}{n}\right), \tag{2.1}$$

while the difference between the two sums in (2.1) is

$$\frac{1}{n}\left\{f\left(\frac{1}{n}\right)-f(1)\right\}. \tag{2.2}$$

That the limit of (2.2), as $n\to\infty$, is zero, follows from the well known result that if $f(x)$ steadily increases as x decreases and $\oint_0^1 f(x)dx$ is convergent, then $\lim_{x\to 0+} xf(x)=0$. From this the conclusion is immediate.

It appears that the following extension of the above theorem is new, although the theorem is quite elementary.

THEOREM 2.2. *Let $f(x)$ be positive and tend steadily to ∞ as x tends to zero. Let $\oint_0^1 f(x)dx$ be convergent. Let $(1/Mn)\leqq\xi_1^{(n)}\leqq(1/n)$ ($M\geqq 1$ and fixed), and $(k-1)/n\leqq\xi_k^{(n)}\leqq(k/n)$ ($k=2, 3, \cdots, n$). Then*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f(\xi_k^{(n)}) = \oint_0^1 f(x)dx.$$

PROOF. Define

$$A_n \equiv \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)$$

and

$$\begin{aligned} B_n &\equiv \frac{1}{n}\left\{f\left(\frac{1}{Mn}\right)+\sum_{k=2}^{n} f\left(\frac{k-1}{n}\right)\right\} \\ &\equiv \frac{1}{n}\left\{f\left(\frac{1}{Mn}\right)+\sum_{k=1}^{n} f\left(\frac{k}{n}\right)-f(1)\right\}. \end{aligned}$$

Then

$$A_n \leqq \frac{1}{n} \sum_{k=1}^{n} f(\xi_k^{(n)}) \leqq B_n.$$

But, clearly, $\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n = \mathscr{P}_0^1 f(x)dx$, and our conclusion follows.

A sharper criterion, in order that the relation

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \oint_0^1 f(x)dx \tag{2.3}$$

should hold whenever the improper integral on the right is convergent, has been given by Wintner [3] in the following result:

THEOREM 2.3. *If $f(x)$ is of bounded variation on every interval $\epsilon \leqq x \leqq 1$, where $\epsilon > 0$, and behaves, as $x \to 0$, so as to satisfy the restriction*

$$\int_\epsilon^1 \left| df(x) \right| = o(\epsilon^{-1}),$$

then the convergence of $\mathscr{P}_0^1 f(x)dx$ implies that

$$\lim_{\epsilon\to 0+} \epsilon \sum_{k\epsilon \leqq 1} f(k\epsilon) \tag{2.4}$$

exists and equals $\mathscr{P}_0^1 f(x)dx$.

The result (2.3) would follow from (2.4) by letting $\epsilon = 1/n$.

The following extension of Wintner's result appears to be new:

THEOREM 2.4. *Let $f(x)$ be of bounded variation on every interval $\epsilon \leqq x \leqq 1$, where $\epsilon > 0$, and let $f(x)$ behave, as $x \to 0^+$, so as to satisfy the restriction*

$$\int_\epsilon^1 \left| df(x) \right| = o(\epsilon^{-1}). \tag{2.5}$$

Let $\xi_1 = 1/Mn$ ($M \geqq 1$ and fixed), and $(k-1)/n \leqq \xi_k^{(n)} \leqq (k/n)$ ($k = 2, 3, \cdots, n$). Let $\mathscr{P}_0^1 f(x)dx$ exist.

Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(\xi_k^{(n)}) = \oint_0^1 f(x)dx.$$

PROOF. From the hypothesis it follows that

$$\lim_{x\to 0} xf(x) = 0. \tag{2.6}$$

For,

$$\epsilon \left| f(1) - f(\epsilon) \right| = \epsilon \left| \int_{\epsilon}^{1} df(x) \right| \leqq \epsilon \int_{\epsilon}^{1} \left| df(x) \right| \to 0.$$

From Wintner's result, we know

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \oint_{0}^{1} f(x)dx.$$

Consider

$$\left\{ \frac{1}{n} \sum_{k=1}^{n} f(\xi_k^{(n)}) - \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \right\}.$$

For all n,

$$\left| \sum_{k=1}^{n} f(\xi_k^{(n)}) - \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \right| \leqq \sum_{k=1}^{n} \left| f(\xi_k^{(n)}) - f\left(\frac{k}{n}\right) \right|$$
$$< \int_{(1/n)}^{1} \left| df(x) \right| + \left| f\left(\frac{1}{Mn}\right) \right| + \left| f\left(\frac{1}{n}\right) \right|.$$

Hence, using (2.5) and (2.6),

$$\lim_{n\to\infty} \left| \frac{1}{n} \sum_{k=1}^{n} f(\xi_k^{(n)}) - \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \right| = 0,$$

and our conclusion follows.

It should be noted that a number of theorems are available in the "converse" direction, that is, in which the convergence of the improper integral $\mathscr{P}_0^1 f(x)dx$ follows from the knowledge of the existence of $\lim_{\epsilon\to 0+} \epsilon \sum_{n\epsilon\leqq 1} f(n\epsilon)$. In this paper, however, we are not concerned with such results.

3. **A pattern integral theorem.** To hope for a complete extension of Theorem 1.1 to improper integrals of the form $\mathscr{P}_0^1 f(x)dx$ is too much. Counter examples are not difficult to construct. For example, see §5 of the paper by Bromwich and Hardy [2].

The following theorem partially extends the earlier results.

THEOREM 3.1. *Let $f(x)$ be positive and tend steadily to ∞ as x tends to zero. Let $\mathscr{P}_0^1 f(x)dx$ be convergent. Let $(1/Mn) \leqq \xi_1^{(n)} \leqq (1/n)$ ($M \geqq 1$ and fixed), and $(k-1)/n \leqq \xi_k^{(n)} \leqq k/n$ ($k = 2, 3, \cdots, n$). Let the pattern*

P be characterized by the dyadic number

$$t = 0.\alpha_1\alpha_2\alpha_3 \cdots \alpha_n \cdots (2),$$

such that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \alpha_k = \alpha.$$

Then

$$(P)\oint_0^1 f(x)dx \equiv \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \alpha_k f(\xi_k^{(n)}) = \alpha\oint_0^1 f(x)dx.$$

PROOF. Define $f_N(x)=f(1/MN)$, $0\leqq x<1/N$, $f_N(x)=f(x)$, $1/N \leqq x\leqq 1$. For all values of $N>0$, $(R)\int_0^1 f_N(x)dx$ exists and

$$(3.1) \qquad (P)\int_0^1 f_N(x)dx = \alpha(R)\int_0^1 f_N(x)dx.$$

Consider the expression

$$(3.2) \qquad \frac{1}{n}\sum_{k=1}^{n} \alpha_k f(\xi_k^{(n)}).$$

Let the subinterval in which $1/N$ occurs be the $\lambda(n)$th. (Then $\lim_{n\to\infty} \lambda(n)/n=1/N$.) Hence (3.2) can be written as $\mathrm{I}_n+\mathrm{II}_n-\mathrm{III}_n+\mathrm{IV}_n$, where

$$\mathrm{I}_n = \frac{1}{n}\alpha_{\lambda(n)} f(\xi_{\lambda(n)}^{(n)}), \quad \mathrm{II}_n = \frac{1}{n}\sum_{k=1}^{n} \alpha_k f_N(\xi_k^{(n)}),$$

$$\mathrm{III}_n = \frac{1}{n}\sum_{k=1}^{\lambda(n)} \alpha_k f_N(\xi_k^{(n)}), \quad \mathrm{IV}_n = \frac{1}{n}\sum_{k=1}^{\lambda(n)-1} \alpha_k f(\xi_k^{(n)}).$$

Let us consider the behavior of I_n, II_n, III_n, and IV_n, as $n\to\infty$. Clearly $f(\xi_{\lambda(n)}^{(n)})\leqq f(1/Mn)$, and since $\lim_{x\to 0+} xf(x)=0$, $\lim_{n\to\infty} \mathrm{I}_n=0$. From (3.1), we see that

$$\lim_{n\to\infty} \mathrm{II}_n = \alpha(R)\int_0^1 f_N(x)dx = \alpha\left\{\frac{1}{N}f\left(\frac{1}{MN}\right) + (R)\int_{1/N}^1 f(x)dx\right\}.$$

Now

$$\mathrm{III}_n = f\left(\frac{1}{MN}\right)\frac{\lambda(n)-1}{n}\,\frac{1}{\lambda(n)-1}\sum_{k=1}^{\lambda(n)-1} \alpha_k + \frac{1}{n}\alpha_{\lambda(n)} f_N(\xi_{\lambda(n)}^{(n)}).$$

Hence,

$$\lim_{n\to\infty} \mathrm{III}_n = \alpha \frac{1}{N} f\left(\frac{1}{MN}\right).$$

We have

$$\mathrm{IV}_n \leqq \frac{1}{n} \sum_{k=1}^{\lambda(n)-1} f(\xi_k^{(n)}).$$

Hence

$$0 \leqq \mathrm{IV}_n \leqq \oint_0^{1/N} f(x)dx + \frac{1}{n} f\left(\frac{1}{Mn}\right)$$

for all n. We then see that, as $n\to\infty$, the expression (3.2) at worst oscillates between

$$\alpha(R) \int_{1/N}^{1} f(x)dx \tag{3.3}$$

and

$$\alpha(R) \int_{1/N}^{1} f(x)dx + \oint_0^{1/N} f(x)dx. \tag{3.4}$$

But this is true for all $N>0$, so letting $N\to\infty$, we see from (3.3) and (3.4) that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \alpha_k f(\xi_k^{(n)}) = \alpha(R) \oint_0^1 f(x)dx,$$

which is the desired result.

4. **Miscellaneous results.** The following is an extension of Theorem 3.1.

THEOREM 4.1. *Let $\mathscr{P}\int_0^1 f(x)dx$ be convergent. Let there be a $g(x)$ with the following properties:*

1. *In the interval $0<x\leqq r$, where $0<r\leqq 1$, $g(x)$ is positive and tends steadily to ∞ as x tends to zero.*
2. *$\mathscr{P}\int_0^r g(x)dx$ is convergent.*

Let $|f(x)| \leqq g(x)$, $0<x\leqq r$. Let $1/Mn \leqq \xi_1^{(n)} \leqq 1/n$ ($M\geqq 1$ and fixed), and $(k-1)/n \leqq \xi_k^{(n)} \leqq k/n$ ($k=2, 3, \cdots, n$). Let the pattern P be characterized by the dyadic number

$$t = 0.\alpha_1\alpha_2\alpha_3 \cdots \alpha_n \cdots (2),$$

such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \alpha_k = \alpha.$$

Then,

$$(P) \oint_0^1 f(x)dx \equiv \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \alpha_k f(\xi_k^{(n)}) = \alpha \oint_0^1 f(x)dx.$$

Since the proof is similar to that of Theorem 3.1, it will be omitted. It is of interest to note that if one chooses

$$f(x) = \frac{1}{x} \cos \frac{1}{x},$$

then $\oint_0^1 f(x)dx$ converges, but

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(\xi_k^{(n)}) \tag{4.1}$$

fails to exist.

Define

$$A_n = \frac{1}{n} f\left(\frac{1}{Mn}\right) + \frac{1}{n} \sum_{k=2}^{n} f\left(\frac{k-1}{n}\right) \qquad (M \geqq 1 \text{ and fixed}),$$

$$B_n = \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right).$$

Then $A_n - B_n = (1/n)\{f(1/Mn) - f(1)\}$. If (4.1) exists,

$$\lim_{n\to\infty} M \frac{1}{Mn} f\left(\frac{1}{Mn}\right) = 0,$$

which is certainly not the case when $f(x) = (1/x) \cos (1/x)$.

Bibliography

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