1. Introduction. A linearly connected space $L_n$ is characterized by its coefficients of (asymmetric) connection $L_{jk}^i$ which transform according to the law [2, p. 3]

$$L_{jk}^i(x) \frac{\partial x^a}{\partial x'^j} = L_{bc}^a(\bar{x}) \frac{\partial x^b}{\partial x'^j} \frac{\partial x^c}{\partial x'^k} + \frac{\partial^2 x^a}{\partial x'^j \partial x'^k}. \tag{1.1}$$

An $L_n$ will admit an infinitesimal motion defined by the infinitesimal transformation

$$\bar{x}^i = x^i + \xi^i(x) \delta t \tag{1.2}$$

if

$$\frac{\partial \xi^i}{\partial x'^j} + \Gamma^i_{jk} \frac{\partial \xi^j}{\partial x'^k} + \Gamma^i_{k} \frac{\partial \xi^k}{\partial x'^j} = 0, \tag{1.3a}$$

$$\frac{\partial \Omega^i_{jk}}{\partial x'^j} + \Omega^i_{hk} \frac{\partial \xi^j}{\partial x'^k} + \Omega^i_{kh} \frac{\partial \xi^h}{\partial x'^k} = 0, \tag{1.3b}$$

where

$$L_{jk}^i = \Gamma_{jk}^i + \Omega_{jk}^i \tag{1.4}$$

and $\Gamma_{jk}^i$, $\Omega_{jk}^i$ are the symmetric and skew-symmetric parts respectively of $L_{jk}^i$ [1, p. 231].

In case $\Omega_{jk}^i = 0$, the $L_n$ reduces to a symmetrically connected space $A_n$ in which (1.3a) defines the infinitesimal affine collineations [2, p. 125], so that motions in an $L_n$ can be considered as generalizations of such collineations of an $A_n$ [3].

In a previous paper [3] we obtained all (symmetric) two-dimensional spaces $A_2$ admitting real continuous groups of affine collineations, these being obtained by solving (1.3a) for the $\xi^i$ for the $\xi^i$ being known and obtained from Lie's classification [4, pp. 71–73, 379–380] giving all real continuous groups $G_r$ in two variables. In §2 we carry through a like procedure to obtain all linearly connected $L_2$ admitting complete groups of motions. This will involve the solution of the
combined system (1.3a–b) for the $\Gamma_{jk}^i$ and $\Omega_{jk}^{ab}$. The results of [3] will of course eliminate much of the analysis which would otherwise be necessary. We shall exclude all solutions of the system (1.3) which lead to $A_z$ spaces ($\Omega_{jk}^{ab} = 0$).

In §3 we consider the special case of spaces of absolute parallelism $T_n$, and determine all $T_2$ admitting motion groups. The conditions on an $L_n$ to reduce to a $T_n$ are [1, p. 234]

(1.5) \[ L_{ijkm} = \frac{\partial L_{ijm}^i}{\partial x^k} - \frac{\partial L_{ijk}^i}{\partial x^m} + L_{ijm}^h L_{hk}^i - L_{ijk}^h L_{hm}^i = 0, \]

that is, the $L_n$ is of zero curvature. In this case there exists an ennuple of (absolutely) parallel vector fields $h_{a_1}^i$, that is,

(1.6) \[ h_{a_1}^i = \frac{\partial h_{a_1}^i}{\partial x^i} + h_{a_1}^m L_{mi}^j = 0 \quad (\alpha = 1, \ldots, n). \]

These equations (1.6) are solved for the $h_{a_1}^i$ using for the $L_{ij}$ the values found from the solutions of (1.3). (The integrability conditions of (1.6) are (1.5).)

The problem of motions in a $T_n$ has been treated by Robertson [5] who obtained the conditions for infinitesimal motions of a $T_n$ in the form

(1.7) \[ \frac{\partial h_{a_1}^i}{\partial x^j} \xi^j - h_{a_1}^i \frac{\partial \xi^i}{\partial x^j} = \epsilon_a h_{\beta_1}, \]

where $\epsilon_a$ are constants. (See also [6].)

It is shown in §2 that there are 8 possible types of $L_2$ admitting groups of motions, the maximum number of parameters being four. In §3 it is shown there are 7 types of motions for a $T_2$.

2. Motions in an $L_2$. We proceed to the solution of (1.3) for the case $n=2$. It is easily shown that (1.3b) reduces to

(2.1a) \[ \frac{\partial A}{\partial x} \xi^1 + \frac{\partial A}{\partial y} \xi^2 + A \frac{\partial \xi^2}{\partial y} - B \frac{\partial \xi^1}{\partial y} = 0, \]

(2.1b) \[ \frac{\partial B}{\partial x} \xi^1 + \frac{\partial B}{\partial y} \xi^2 + B \frac{\partial \xi^1}{\partial x} - A \frac{\partial \xi^2}{\partial x} = 0, \]

where

(2.2) \[ A = \Omega_{12}^1, \quad B = \Omega_{12}^2 \quad (x, y = x^1, x^2). \]
Corresponding to each $G_r$ of Lie's classification we solve (2.1) for $A$ and $B$, excluding all solutions for which $A = B = 0$. (The solutions of (1.3a) for the $\Gamma_{jk}^i$ have been obtained in [3].) For each of the nonexcluded solutions the $G_r$ is tested for completeness as explained in [3].

As an illustration we consider the $G_3[p, q, xq]$ where $p = \frac{\partial f}{\partial x}$, $q = \frac{\partial f}{\partial y}$. Using $X_1f = p$, $X_2f = q$, $X_3f = xq$, with $\xi_1^i = \delta_1^i$, $\xi_2^i = \delta_2^i$, $\xi_3^i = 0$, $\xi_3^i = x$ in (2.1) we obtain $A = 0$, $B = B_0$ an arbitrary constant.

To obtain the complete group, put $A = 0$, $B = B_0$ in (2.1), and solve for $\xi_i$. This gives $\xi_1^i = k$, $\xi_2^i = \xi_2^i(x, y)$, with $k$ an arbitrary constant and $\xi_2^i$ an arbitrary function. Now as determined in [3] the $\Gamma$'s obtained as solutions of (1.3a) corresponding to the above $G_3$ will admit as their complete group the $G_4[p, q, xq, yq]$. It is thus seen that by taking $k = 0$, $\xi_2^i = y$, we obtain the fourth generator $yq$ of the $G_4$, and hence this $G_4$ will be the complete group of motions of the $L_2$ given by

\[
L_{11} = 2a, \quad L_{12} = L_{21} = L_{12}^1 = L_{22}^1 = 0, \quad L_{22}^2 = a + B_0, \quad L_{21}^2 = a - B_0 \quad (B_0 \neq 0).
\]

The $\Gamma$'s used to give these $L$'s are obtained from group [4.1] of [3].

By proceeding as above with each $G_r$ and making use of the results of [3] we find eight possible types of $L_2$ spaces admitting real groups of motions. These with their respective groups are given below.

Asymmetrically connected space $L_2$ and their complete groups of motions

[A1.1] $[p]$ $L_{jk}^i = L_{jk}^i(y)$ (arbitrary functions of $y$).

[A2.1] $[p, q]$ $L_{jk}^i = \text{arbitrary constants}.$

[A2.2] $[p, xp + yq]$ $L_{jk}^i = (1/y)A_{jk}^i \quad (A_{jk}^i \text{ arbitrary constants}).$

[A3.1] $[p, 2xp + yq, x^2p + xyq]$ $L_{11}^i = \frac{2b}{y^2}, \quad L_{12}^i = -\frac{1}{y}, \quad L_{21}^i = -\frac{1}{y}, \quad L_{22}^1 = 0,$

$L_{11}^2 = \frac{a}{y^2}, \quad L_{12}^2 = \frac{b + c}{y^2}, \quad L_{21}^2 = \frac{b - c}{y^2}, \quad L_{22}^2 = -\frac{2}{y} \quad (c \neq 0).$
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[A3.2] \[q, xq, yq\]
\[L_{11} = 2f(x), \quad L_{12} = L_{21} = L_{22} = L_{11} = L_{22} = 0,\]
\[L_{23} = f(x) + B(x), \quad L_{21} = f(x) - B(x) \quad (B \neq 0).\]

[A4.1] \[p, q, xq, yq\]
\[L_{11} = 2a, \quad L_{12} = L_{21} = L_{22} = L_{11} = L_{22} = 0,\]
\[L_{23} = a + b, \quad L_{21} = a - b \quad (b \neq 0).\]

[A4.2] \[p, q, yq, e^q\]
\[L_{11} = 2a, \quad L_{12} = L_{21} = L_{22} = L_{11} = L_{22} = 0,\]
\[L_{23} = a + b, \quad L_{21} = a - b \quad (b \neq 0).\]

[A4.3] \[p, yq, (e^q \cos x)q, (e^q \sin x)q\]
\[L_{11} = 1/3 + 2a, \quad L_{12} = L_{21} = L_{22} = L_{11} = L_{22} = 0,\]
\[L_{23} = a + c - 1/3, \quad L_{21} = a - c - 1/3 \quad (c \neq 0).\]

In the above \(a, b, c\) are arbitrary constants, \(f(x), B(x)\) are arbitrary functions subject to indicated restrictions.

3. Motions in spaces \(T_2\) of absolute parallelism. For each of the 8 types of \(L_2\) obtained in the previous section, we impose conditions (1.5), and then solve (1.6) for the \(h_{a1}\).

The case [A3.2] will be used as an illustration. Conditions (1.5) reduce to \((f + B)' - (f + B)^2 = 0\), so that \(f(x) + B(x) = -1/(x + c)\) with \(c\) an arbitrary constant. Equations (1.6) are

\[
\frac{\partial h_1}{\partial x} + 2h_1f(x) = 0, \quad \frac{\partial h_2}{\partial x} + h_2(f(x) - B(x)) = 0,
\]

\[
\frac{\partial h_1}{\partial y} = 0, \quad \frac{\partial h_2}{\partial y} - \frac{h_2}{x + c} = 0,
\]

which can be solved to give \(h_1 = ag(x), \quad h_2 = (ay + b)g(x)/(x + c)\), where \(g(x) = \exp \{-2f(x)dx\}\).

The complete solution is then

\[
h_{a1} = a_ag(x), \quad h_{a2} = \frac{a_ay + b_a}{x + c}g(x) \quad (\alpha = 1, 2),
\]

with \(a_a, b_a, c\) arbitrary constants such that \(a_1b_2 - a_2b_1 \neq 0\). (The special case \(L_{12}^2 = f + B = 0\) gives the solution \(h_{a1}^1 = a_ag(x), \quad h_{a2}^2 = b_ag(x)\).)
It is of interest to verify that the $T_2$ will admit the finite group generated by the infinitesimal group. To show this we first obtain the finite equations of the group, these being

\[(3.3) \quad \ddot{x} = x, \quad \ddot{y} = \lambda x + \mu y + \nu \]

with the 3 parameters $\lambda$, $\mu$, $\nu$.

Now we must have [5]

\[(3.4) \quad h^i_\alpha (\dot{x}, \dot{y}) = A^\alpha_i \delta^i_\beta (x, y) \frac{\partial \xi^\beta}{\partial x^i}, \]

where the $A^\alpha_i$ must be functions $A^\alpha_\beta (\lambda, \mu, \nu)$ of the parameters only. From (3.2), (3.3), and (3.4) we obtain the relations

\[(3.5) \quad A^\alpha_a c_\beta = a_\alpha, \quad A^\alpha_a b_\beta = c_\alpha, \]

where $c_\alpha = [a_\alpha (\nu - \lambda \xi) + b_\alpha] / \mu$. It follows that the $A^\alpha_\beta$ exist and are of the required form.

The results from the solutions of (1.6) are given below.

**Spaces $T_2$ of absolute parallelism admitting complete groups of motions**

[T1.1] [5]

(a) $h^1_{a1} = a^i_\alpha \exp \left\{ \int (\lambda L^1_{2i} - L^1_{1i})dx^i \right\}$,

$h^2_{a1} = -\sum_i a^i_\alpha \lambda^i_\alpha \exp \left\{ \int (\lambda L^1_{2i} - L^1_{1i})dx^i \right\};$

(b) $h^1_{a1} = \exp \left\{ \int (\lambda L^1_{2i} - L^1_{1i})dx^i \right\} \left[ a_\alpha (\lambda \cos \rho + b_\alpha \sin \rho) \right.$

$\left. + b_\alpha (\lambda \sin \rho + \mu \cos \rho) \right]$;

(c) $h^1_{a1} = \exp \left\{ \int (\lambda L^1_{2i} - L^1_{1i})dx^i \right\} \left[ a_\alpha \int \frac{L^1_{2i}dx^i}{L^1_{1i}} + b_\alpha \right]$,

$h^2_{a1} = -\exp \left\{ \int (\lambda L^1_{2i} - L^1_{1i})dx^i \right\} \left[ \frac{a_\alpha}{L^1_{2i}} + \lambda \left( a_\alpha \int \frac{L^1_{2i}dx^i}{L^1_{1i}} + b_\alpha \right) \right].$

In (a), $\lambda_1 \neq \lambda_2$ are the real roots of $f(\lambda) \equiv \lambda^2 L^1_{21} + \lambda (L^2_{21} - L^1_{11}) - L^1_{11} = 0$;
in (b), \( \lambda \pm i\mu \) are its roots; and in (c), \( f(\lambda) \) is assumed to have equal roots, \( \lambda, \lambda \). In (b), \( \rho = \int_\mu l_{ij}^2 dx^i \). The \( L_{ij}^k(y) \) are otherwise arbitrary subject to (1.5), which have as a consequence that \( L_{11} + L_{21} \) and \( L_{11}L_{12} - L_{21}L_{11} \) must be constant. The \( a_a, b_a, a_a' \) are arbitrary constants such that \( h_{a1}^i \) are linearly independent. For special values of the \( L_{ij}^k \), for example, \( L_{21}^1 = 0 \), the solutions for the \( h_{a1}^i \) can be easily obtained.

[T2.1] \( \{p, q\} \)

The \( h_{a1}^i \) are obtained as in the 3 cases of [T1.1] except the \( L_{ij}^k \) are now considered constants.

[T2.2] \( \{p, xp + yq\} \)

\[
\begin{align*}
h_{a1}^1 &= y^m (a_a u + b_a), \\
h_{a1}^2 &= y^m (c_v + d_a),
\end{align*}
\]

with \( c_v A_{11}^1 + a_v A_{11}^2 = 0, d_v A_{21}^1 + b_v A_{21}^2 = 0 \), \( m = -(1 + A_{21}^1 + A_{22}^2)/2 \), and \( u = xA_{11}^2 + yA_{21}^2, v = xA_{11}^2 + yA_{21}^2 \). The constants \( A_{ij}^k \) are arbitrary subject to (1.5). As in [T1.1] the solutions for \( h_{a1}^i \) corresponding to special values of the \( A_{ij}^k \) can be obtained without difficulty.

[T3.1] \( \{p, 2xp + yq, x^2p + xyq\} \)

\[
\begin{align*}
h_{a1}^1 &= y(a_a x + b_a), \\
h_{a1}^2 &= (y^2 + 2bx)a_a + 2bb_a.
\end{align*}
\]

The \( a, b, c \) of [A3.1] must satisfy \( c = 3b, a = 4b^2, b \neq 0 \).

[T3.2] \( \{q, xy, yq\} \)

\[
\begin{align*}
h_{a1}^1 &= a_v g(x), \\
h_{a1}^2 &= (a_v y + b_v)g(x)/(x + c).
\end{align*}
\]

Here the \( L_{ik}^j \) of [A3.2] satisfy \( g(x) = \exp \{-2f(x)dx\} \), and \( f(x) + B(x) = -1/(x + c) \).

[T4.1] \( \{p, q, xq, yq\} \)

\[
\begin{align*}
h_{a1}^1 &= a_v e^{-2ax}, \\
h_{a1}^2 &= b_v e^{-2ax},
\end{align*}
\]

with \( a + b = 0, (b \neq 0) \) in [A4.1].

[T4.2] \( \{p, q, yq, e^yq\} \)

\[
\begin{align*}
h_{a1}^1 &= a_v e^{(3x-1)x}, \\
h_{a1}^2 &= b_v e^{3x} \\
h_{a1}^1 &= a_v e^{(3x+1)x}, \\
h_{a1}^2 &= (a_v y + b_v) e^{(3x+1)x},
\end{align*}
\]

with \( a + c = 1/3 \) in [A4.2], \( a + c = -2/3 \) in [A4.2].

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