AN ELEMENTARY PROOF OF THE JORDAN-SCHOENFLIES THEOREM[1]

STEWART S. CAIRNS

1. Introduction.

THE JORDAN-SCHOENFLIES THEOREM. *A simple closed curve c in a plane E separates E into two regions. There exists a self-homeomorphism of E under which c is mapped onto a circle.*

The *exterior* of a bounded closed point set b in E will mean the unbounded region of the complementary set $E-b$. The remainder of $E-b$, if not vacuous, will be called the *interior* of b.

(A) *As a corollary to the above theorem, c is intersected by any simple arc with one end point interior and one exterior to c.*

This paper contains an elementary constructive proof of the Jordan-Schoenflies Theorem, motivated by the belief that such a proof should be presented at a fairly early stage to students of topology and analysis. To that end, it is desirable that the argument be disassociated from conformal mapping theory and be accomplished by methods as elementary as possible.

2. **Preliminary results.** Let (x, y) denote a rectangular cartesian coordinate system in E. The following two statements can be quickly established by familiar methods.

(A) *Let b_1 and b_2 denote two simple closed curves for each of which the Jordan-Schoenflies Theorem holds. Then an arbitrary homeomorphism between b_1 and b_2 can be extended to a self-homeomorphism of E.*

(B) If the Jordan-Schoenflies Theorem holds for b_1 and b_2, and if the intersection $b_1 \cdot b_2$ is a simple arc b, then the Jordan-Schoenflies Theorem holds for the simple closed curve b_1+b_2-b', where b' denotes b without its end points.

THEOREM 2.1. *The Jordan-Schoenflies Theorem holds for a simple closed polygon p. A polygonal path crossing p at just one point and otherwise not meeting p has one end point exterior and one interior to p.*

PROOF. (C) The result offers no difficulty when p is a triangle.

Presented to the Society, December 30, 1948; received by the editors June 5, 1950 and, in revised form, January 15, 1951.

[1] The proof has been substantially shortened and simplified since the presentation of the paper to the Society as a result of suggestions by Mr. John Nash of Princeton University.

Suppose p has $n>3$ vertices and assume Theorem 2.1 for all polygons having fewer than n vertices.

Let α be the set of all points each attainable from the exterior of p by a polygonal path crossing p at just one point and otherwise not meeting p.

LEMMA 2.1. *There exists a line segment d joining*[2] *two vertices of p on α.*

To establish Lemma 2.1, let P_0 be the point on p with the smallest ordinate among those where the absicssa is smallest. Then P_0 is a vertex of p. Let P_1, P_2 be the vertices consecutive with P_0 in either sense along p. Let δ denote the triangular region $P_0P_1P_2$. Then either P_1P_2 satisfies Lemma 2.1 or else $\bar{\delta}$ contains vertices of p other than (P_1, P_2). In the latter case, P_0P_3 satisfies the lemma if P_3 is one of the vertices on $\bar{\delta}-(P_1, P_2)$ with least abscissa greater than the abscissa of P_0.

Let p_1, p_2 be the two polygonal arcs into which the end points of d divide p. Then the hypothesis of the recurrency (see (A) above) applies to p_1+d and to p_2+d. Theorem 2.1 now follows for p, and hence follows in general, with the aid of result (B).

3. Approximation to a sector.

LEMMA 3.1. *Let c be a Jordan curve with at least one interior point P and let α be the maximal region of $E-c$ containing P. Then any chord*

$$d = D_1D_2 \tag{3.1}$$

of c on α separates α into two regions.

PROOF. Let c_1, c_2 be the two arcs into which D_1, D_2 separate c. Let p denote an arbitrary simple closed polygon crossing d at just one point M, and not meeting d elsewhere.

(A) The polygon p intersects c_i $(i=1, 2)$.

This auxiliary result follows from the facts that (1) p separates D_1 from D_2, by Theorem 2.1, and (2) c_i joins D_1 and D_2.

(B) *Let p be traced from M in either sense to the first points encountered on c. This leads to two distinct points, P_1 and P_2, on c_1 and c_2 respectively.*

To establish (B), let p_0 be the arc P_1MP_2 of p. Suppose that (B) is false and that both end points of p_0 are on c_1, for example. Let c_0 be the arc of c_1 which they bound. From parts of p_0 and a suitable

[2] A simple arc will be said to *join* its end points *on a region* if the entire arc, save perhaps for either or both end points, is on that region.

polygonal approximation to c_0, it is possible to put together a simple closed polygon through M, meeting (c_2+d) only at M, where it crosses d. By the argument for (A), this is contradictory, since such a polygon would necessarily meet c_2.

(C) *Let p_i be the arc of p_0 with M and P_i for end points* $(i=1, 2)$, *and let α_i be the set of all points which can be joined to p_i by arcs not meeting $c+d$. Then* (1) *α_1 and α_2 are disjoint and* (2) $\bar{\alpha}_1+\bar{\alpha}_2=\bar{\alpha}$.

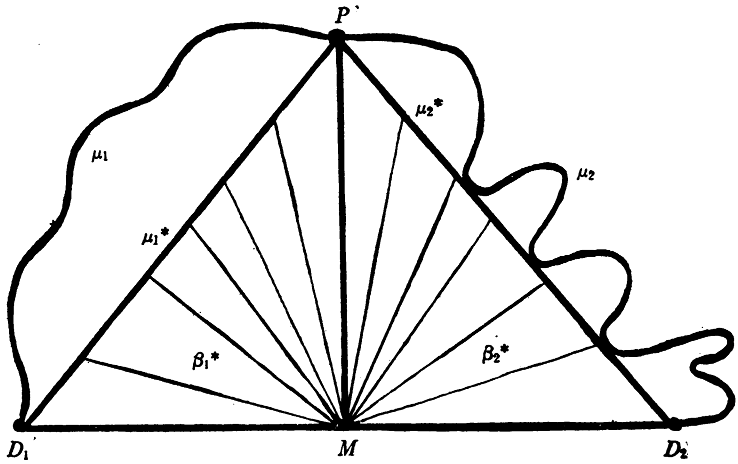

Figure 3.1

If α_1 and α_2 were not disjoint, then any point common to them could be joined to p_i $(i=1, 2)$ by a polygonal arc q_i on α_i. From parts of p_1, p_2, q_1, and q_2, a polygon could be put together, leading to the same sort of contradiction as in the argument for (B). Part (2) of (C) presents no difficulty. The lemma and the corollary below now follow at once.

COROLLARY. *In the above notation, the boundary of α_i is on $d+c_i$* $(i=1, 2)$.

Either of the two parts into which a chord $d=D_1D_2$ separates α will be called a *sector* β of α. As a preliminary to proving that α is a 2-cell, a method will now be developed for partially filling in β by an approximating region β^*. In accordance with the preceding corollary, the boundary of β is on $d+\mu$, where μ is one of the arcs of c with D_1 and D_2 for end points. From the midpoint M of d, let a ray normal to d be extended into β, and let P be the first point of c on that ray. Then, by the above corollary, P is on μ [see Fig. 3.1]. Let μ_i be the arc of μ with P and D_i for end points $(i=1, 2)$. By

Lemma 3.1 and the corollary, MP separates β into two regions β_i $(i=1, 2)$, where the boundary of β_i is on μ_i+MD_i+MP. Let δ_i be the interior of the triangle MPD_i. A subset β_i^* of δ_i will next be defined, as an approximation to β_i. Its boundary will be the union of MP, MD_i, and an arc μ_i^* joining D_i and P on $\bar{\delta}_i$.

Case I. $(\delta_i\subset\beta_i)$. In this case $\mu_i^*=PD_i$ and $\beta_i^*=\delta_i$ (see β_1^* in Fig. 3.1).

Case II. $(\delta_i\not\subset\beta_i)$. In this case, let μ_i' be the intersection of μ_i with δ_i. The convex hull of $\mu_i'+PD_i$ is then bounded by a convex closed curve $\mu_i'+D_iP$; and the arc μ_i^* separates δ_i into two regions, of which the one with M on its boundary will be β_i^* [see β_2^* in Fig. 3.1].

(D) The approximation β^* to β is now defined as the union of β_1^*, β_2^* and the open segment MP. It is uniquely determined by β.

(E) The arc

$$\mu^* = \mu_1^* + \mu_2^* \tag{3.2}$$

is the union of a subset c^* of μ and a denumerable set of chords of c. As a point P^* traces μ^* from D_1 to D_2, the open segment MP^* sweeps out the entire region β^*.

4. **An interior region of a Jordan curve.** Under the hypotheses of Lemma 3.1, let $h(c)$ be an arbitrary but fixed homeomorphic mapping of c onto a circle k. The images of c^*, μ, and D_i [see §3 for notation] will be denoted by

$$\begin{aligned} k^* &= h(c^*), \\ v &= h(\mu), \\ E_i &= h(D_i) \qquad (i = 1, 2). \end{aligned} \tag{4.1}$$

Then v is an arc of k with end points E_1, E_2, and k^* is a subset of v. Let $h(c)$ now be extended to each chord of c on μ^* [see §3(E)] and to the chord d by the requirement that these cords map linearly onto chords of k. This extends $h(c)$ into a map $h(c+d+\mu^*)$. Let the images of d and μ^* [see §3] be

$$\begin{aligned} e &= h(d), \\ v^* &= h(\mu^*). \end{aligned} \tag{4.2}$$

Then e is the chord E_1E_2 and v^* is a simple arc from E_1 to E_2 on the closure of the region γ bounded by $e+v$. The arc v^* is the union of k^* and a denumerable set of chords of k.

The image $N=h(M)$ of M is the midpoint of e. As P^* traces μ^* from D_1 to D_2, its image $Q^*=h(P^*)$ traces v^* from E_1 to E_2, and the open segment NQ^* sweeps out an approximation γ^* to γ, bounded by $e+v^*$. Let h now be extended over β^* by the requirement that it map each segment MP^* linearly onto the segment NQ^*.

(A) This completes the extension of $h(c)$ into a homeomorphic mapping $h(c+\beta^*)$ of $c+\beta^*$ onto $k+\gamma^*$. The extension is uniquely determined, given $h(c)$ and the sector β.

THEOREM 4.1. *Under the hypotheses of Lemma* 3.1, *an arbitrary homeomorphic mapping $h(c)$ of c onto a circle k can be extended into a homeomorphic mapping $H(\bar{\alpha})$ of $\bar{\alpha}$ onto $\overline{K}$, where K is the interior of k.*

Let $d=D_1D_2$ be an arbitrary chord of c on α. Let β^*, γ^* be approximations, as defined in §3(D), to the two sectors into which d separates α. Let α_1 denote the union of β^*, γ^* and the open segment D_1D_2.

(B) *By the extension process of statement* (A), *let $h(c)$ be extended over both $\bar{\beta}^*$ and $\bar{\gamma}^*$, hence over $\bar{\alpha}_1$. This defines a homeomorphic mapping $H_1(\bar{\alpha}_1+c)$ of $\bar{\alpha}_1+c$ onto a certain subset $\overline{K}_1+k$ of $\overline{K}$.*

The definition of $H_1(\bar{\alpha}_1+c)$ is the first step of a recurrent process, based on the following hypothesis, which is easy to verify for $j=1$.

HYPOTHESIS. For some positive integer j, the sets α_i, K_i and the homeomorphisms H_i $(i=1, 2, \cdots, j)$ have been so defined that:

(1) The domain of H_i is $\bar{\alpha}_i+c$, where α_i is a 2-cell on α.

(2) The boundary of α_i is a simple closed curve c_i which is the union of a point set c_i^* on c and a denumerable set of chords of c on α.

(3) The image $k_i=H_i(c_i)$ is the union of a subset $k_i^*=H_i(c_i^*)$ of k and a denumerable set of chords of k.

(4) The mapping H_i is linear between the chords of c on c_i and the chords of k on k_i.

(5) Each map H_{i+1} is an extension of H_i and of $h(c)$.

Let d_i $(i=1, 2, \cdots)$ denote the chords of c on c_j. Of the two sectors into which d_i separates α (see Lemma 3.1), let β_i be the one which contains no point of α_j, and let β_i^* be the approximation to β_i defined in §3(D).

(C) The region α_{j+1} is now defined as

$$\alpha_{j+1} = \alpha_j + \sum \beta_i^* + \sum d_i' \tag{4.3}$$

where d_i' is the chord d_i without its end points. The homeomorphism H_{j+1} is now defined on $\bar{\alpha}_{j+1}-\bar{\alpha}_j$ by extending $h(c)$ over each $\bar{\beta}_*^i$, using the process of (A) above. On c_j, this extension agrees with H_j, as a consequence of the linearity requirements in the extension process and in part (4) of the above hypothesis.

On $\bar{\alpha}_j$, let H_{j+1} be defined as identical with H_j. It is then easy to verify the above Hypothesis with $j+1$ in place of j where

$$K_{j+1} = H_{j+1}(\alpha_{j+1}), \quad k_{j+1} = H_{j+1}(c_{j+1}). \tag{4.4}$$

LEMMA 4.1. *For any $\epsilon > 0$, there exists an integer j so large that every circular region of radius ϵ about a point on c contains a sector on α outside α_j and cut off by a chord of c_j. In other words, the regions of Fig.* 3.1 *become uniformly small as j increases.*

Suppose the lemma false. As a consequence of the recurrent process for defining the regions α_j, it follows that, for some $\epsilon > 0$, there exists at least one circular neighborhood of radius ϵ with center on c containing no point of $\sum c_i^*$.

Let c_0 be a maximal arc of $c - \sum c_i^*$, and let P_0, P_0' be its end points . Let N, N' be the circular neighborhoods about P_0, P_0' respectively each of radius $d(P_0, P_0')/3$. By definition of c_0, it is possible to find two points (D_1, D_2) such that (1) D_1 and D_2 are the end points of a chord d of c on the curve c_i [see Hypothesis, Part (2)] for some value of i. (2) If c_i' is the arc of c which has D_1, D_2 for end points and contains c_0, then $(c_i' - c_0) \subset (N + N')$. (3) D_1 and D_2 are so close to P_0 and P_0' respectively, that the perpendicular bisector n of d does not meet $\overline{N} + \overline{N}'$. In the extension process, a point of c_{i+1}^* is common to n and c_i', hence is on the arc c_0. Since this contradicts the definition of c_0, Lemma 4.1 is proved.

COROLLARY. *Every point of α is on one of the 2-cells α_i.*

Assume the contrary, and let Q be a point on $\alpha - \sum \alpha_i$. For each value of i, there is a chord d_i on c_i which separates α into two sectors, one of which, β_i, contains Q, while the other contains α_i. Let e_i be the arc of c on $\bar{\beta}_i$. Then, as a consequence of Lemma 4.1, there is just one point Q^* common to all the arcs e_i. For i large enough, any given neighborhood $N(Q^*)$ will contain $d_i + c_i$ and hence β_i. Since $N(Q^*)$ need not contain Q, the corollary follows.

Now let H be defined as the common extension of all the homeomorphisms H_i. By the above corollary, the domain of H is $\alpha + c$. Furthermore, H is continuous on $\bar{\alpha}$. Its continuity on α follows from the continuity of the H_i, while its continuity at points on c follows from Lemma 4.1. Hence the mapping H fulfills the requirements of Theorem 4.1.

Let b_1 and b_2 denote two Jordan curves, each with an interior, and let β_i be a 2-cell with b_i for boundary ($i = 1, 2$), in accordance with

Theorem 4.1. Suppose the intersection of b_1 and b_2 is an arc b, where b_1-b is exterior to b_2 and b_2-b is exterior to b_1. Let b' denote b without its end points, and let

$$\beta = \beta_1 + b' + \beta_2. \tag{4.5}$$

(D) *As a corollary to Theorem* 4.1, β *is a 2-cell with* b_1+b_2-b' *for boundary, and any homeomorphism between this boundary and a circle* k *can be extended into a homeomorphism between* $\bar{\beta}$ *and* $\overline{K}$ (*see Theorem* 4.1 *for notation*). *It will be said that* β *is obtained by amalgamating* β_1 *and* β_2 *across* b.

5. Completion of the proof.

LEMMA 5.1. *Let* c *satisfy the hypotheses of Lemma* 3.1 *and hence of Theorem* 4.1. *Let* g *be a simple arc joining two distinct points* P_1 *and* P_2 *of* c *and lying on* α, *save for* P_1, P_2. *Let* c_1, c_2 *be the two arcs into which* P_1, P_2 *divide* c. *Then* g *separates* α *into two 2-cells* α_1, α_2 *where* α_i *has* $g+c_i$ *for boundary* ($i=1, 2$).

PROOF. By Theorem 4.1, the lemma reduces to the case where c is a circle and α is its interior. If g did not separate α, a polygonal arc p could be constructed joining c_1 to c_2 on α without meeting g. This arc p could be completed outside c to a simple closed polygon. By Theorem 2.1, using an arc of circle instead of a polygonal path, such a polygon must separate P_1 from P_2 and hence must intersect g, contrary to its definition. It follows that $g+c_i$ satisfies the hypotheses of Theorem 4.1. Let α_i be a 2-cell bounded by $g+c_i$ in accordance with that theorem. By §4(D), α_1 and α_2 can be amalgamated across g to obtain the 2-cell, α, bounded by c.

Next consider an arbitrary simple closed curve c. Let p be a simple closed polygon, meeting c in just two points, P_1 and P_2, and otherwise exterior to c. Such a polygon is easy to define, if P_1 and P_2 are chosen as points of maximum and minimum ordinates, respectively, on c.

Let c_1 and c_2 be the two arcs into which P_1 and P_2 divide c. As a consequence of Theorem 2.1 and Lemma 5.1, c_i separates the interior, ρ, of p into two 2-cells α_i and β_i, one of which, β_i, contains c_j ($i=1, j=2$) and ($i=2, j=1$). Similarly, c_1 separates β_2 into two 2-cells, one of which, α, has c_1+c_2 for boundary. Any point on α is interior to c, since any arc joining it to p must meet either c_1 or c_2. This establishes the following result.

LEMMA 5.2. *Any Jordan curve has an interior.*

Now let α_1 and α be amalgamated across c_1 [see §4(D)] and let the resulting 2-cell be amalgamated with α_2 across c_2 to obtain a 2-cell ρ with p for boundary. By Theorem 2.1, ρ is the interior of p. Since α_1 and α_2 are exterior to c, the 2-cell α constitutes the entire interior of c. The Jordan-Schoenflies Theorem now follows readily in all its generality.

UNIVERSITY OF ILLINOIS

A NOTE ON CURVATURE AND BETTI NUMBERS

H. GUGGENHEIMER

1. S. Bochner has proved the following theorem [2]:[1] Let $M^{(m)}$ be a closed manifold with complex structure [4; 7] of complex dimension m, on which there exists a Kähler-metric [2; 3; 5][2]

$$ds^2 = g_{ik^*}(dz^i dz^*),^3 \tag{1}$$

$$\frac{\partial g_{ik^*}}{\partial z_l} = \frac{\partial g_{lk^*}}{\partial z_i}. \tag{2}$$

Let R_{ik^*} denote the Ricci tensor and

$$P_{hi^*jk^*} = R_{hi^*jk^*} - \frac{1}{m+1}(g_{hi^*}R_{jk^*} + g_{hk^*}R_{i^*j}) \tag{3}$$

the tensor of projective curvature. In every point of $M^{(m)}$ we form the numbers

$$L = \inf_{\xi} \frac{-R_{ik^*}\xi^i\xi^{k^*}}{\xi^i\xi_i}, \tag{4}$$

$$P = \sup_{\xi} \left| \frac{P_{hi^*jk^*}\xi^{hi^*}\xi^{jk^*}}{\xi^{hi^*}\xi_{hi^*}} \right|, \tag{5}$$

with all vectors ξ^i and skew-symmetric tensors ξ^{ij^*} attached to the point in question. If

$$L > 0 \tag{6}$$

Received by the editors December 8, 1950.

[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] Products of differentials in parentheses denote ordinary products, products without parentheses are skew products.

[3] We denote by i^* the index relative to z^{*i}.