AN EMBEDDING OF PI-RINGS

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1. Introduction. It is well known that a commutative ring which has no nonzero nilpotent ideals is isomorphic to a subring of a complete direct sum of commutative fields (McCoy [1]). In this note, this fact is generalised to rings which satisfy a polynomial identity (PI-rings). We show that every PI-ring which has no nilpotent ideals is isomorphic to a subring of a complete direct sum of central simple algebras whose order over their centre is bounded. As a consequence we prove that these rings are subrings of matrix rings over commutative rings. This implies an extension of a result of [2] concerning the minimal identity of a simple algebra. We prove that for a PI-ring which has no nonzero nilpotent ideals, the standard identity \( S_d(x) = 0 \), where \( d \) is an even integer, is the unique (up to a numerical factor) minimal identity which is linear in each of its indeterminates. The term standard identity was ascribed in [2] to the polynomial identity:

\[
S_d(x) = S_d(x_1, \ldots, x_d) = \sum_{(i)} \pm x_{i_1} \cdots x_{i_d} = 0
\]

where the sum ranges over all permutations \((i)\) of \(d\) letters, and the sign is positive for even permutations and negative for odd permutations.

Notations. A polynomial identity of minimum degree satisfied by a PI-ring \(R\) will be called a minimal identity of \(R\). We shall refer to a polynomial identity which is linear and homogeneous in each of its indeterminates as a linear identity. We shall use the following three types of semi-simplicity: a ring \(R\) is said to be

(a) J-semi-simple, if \(R\) is semi-simple in the sense of Jacobson [3], that is, if the quasi-regular radical of \(R\) is zero.

(b) K-semi-simple, if \(R\) does not contain any nonzero nil ideals.

(c) A-semi-simple, if \(R\) has no nonzero nilpotent ideals.

2. The ring \(R[x]\). We denote by \(R[x]\) the ring of all polynomials in the commutative indeterminate \(x\) over \(R\). In this section we deal with properties of \(R[x]\) induced by \(R\).

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1 Numbers in brackets refer to the bibliography at the end of the paper.

2 Ideals will always mean two-sided ideal.

3 For definition of (complete) direct sums and of subdirect sums see, for example, [1, p. 121].
**Lemma 1.** Let $P$ be a nonzero ideal in $R[x]$ and let $p(x) = a_0 + \cdots + a_n x^n$ ($a_n \neq 0$) be a polynomial of minimum degree in $P$. Then if $b \in R$ such that $a_n b = 0$ for some integer $p$, then $a_n^{-1} p(x) b = 0$.

Indeed, the coefficient of $x^n$ in $a_n^{-1} p(x) b \in P$ is $a_1 b = 0$, that is, this polynomial is of lower degree than that of $p(x)$. Hence the minimality of the degree of $p(x)$ implies that $a_n^{-1} p(x) b = 0$.

**Corollary.** If $r(x) \in R[x]$ such that $a_n^\mu r(x) = 0$ for some integer $\mu$, then $a_n^{\lambda} r(x) = 0$ for every integer $\lambda \geq \mu - 1$.

This follows immediately by the preceding lemma, since each of the coefficients of $r(x)$ satisfies the condition of that lemma.

We prove now the following fundamental lemma:

**Lemma 2.** If $R$ is a K-semi-simple ring, then $R[x]$ is J-semi-simple.

**Proof.** Assume that $R[x]$ is not J-semi-simple. Denote by $J_x$ the nonzero Jacobson’s radical of $R[x]$. It is readily verified that the totality of the coefficients of the highest power of the polynomials of $J_x$ of degree $n$—where $n$ is the minimal degree of the nonzero polynomials of $J_x$—constitute a nonzero ideal in $R$. The lemma will be proved if it is shown that this ideal is a nil ideal, that is, that if $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ is a nonzero polynomial of minimum degree in $J_x$, then $a_n^\mu = 0$ for some integer $\mu$.

To this end we consider the polynomial $p(x) x a_n$ (which belongs to $J_x$, since $p(x) \in J_x$ and $x a_n \in R[x]$) and its quasi-inverse $q(x)$. By Lemma 1 of [3] and Theorem 2 of [3] it follows that

\begin{align*}
(1) & \quad p(x) x a_n + q(x) + p(x) x a_n q(x) = 0, \\
(2) & \quad p(x) x a_n + q(x) + q(x) p(x) x a_n = 0.
\end{align*}

By (1) we obtain that $q(x) = xt(x)$, $t(x) \in R[x]$. Put $s(x) = p(x) a_n$. Then (1) implies that $x s(x) + xt(x) + x^2 s(x) t(x) = 0$. Hence,$^6$

\begin{equation}
(3) \quad s(x) + t(x) + x s(x) t(x) = 0.
\end{equation}

Similarly, we obtain from (2) that

\begin{equation}
(4) \quad s(x) + t(x) + x t(x) s(x) = 0.
\end{equation}

Suppose $a_n^\mu t(x) \neq 0$ for every integer $\mu$. Let $\nu$ be the minimal degree of the polynomials $a_n^\mu t(x)$. Write

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$^4$ If $R$ is commutative, this lemma is a consequence of [7, Corollary 8.1].

$^5$ If $R$ does not possess a unit and $x \in R[x]$, we adopt the notation $xt(x)$ (similarly $t(x)x$) for the polynomial $x b_0 + \cdots + x^n b_n$, where $t(x) = b_0 + \cdots + x^n b_n$.

$^6$ Since $xm(x) = 0$ if and only if $m(x) = 0$. 

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(5) \( t(x) = t_1(x) + x^{r+1}t_2(x) \),
where \( t_1(x) = b_0 + b_1x + \cdots + b_rx^r \). The minimality of \( \nu \) implies that

(6) \( a_n b_\nu \neq 0 \) for every integer \( \mu \),

and

(7) \( a_n^\mu t_2(x) = 0 \) for every \( \mu \) greater than some integer \( \pi \).

The polynomial \( s(x) = s(x)a_n \) is of minimum degree in \( J_x \), and its highest coefficient is \( a_n^\lambda \). Hence, since \( a_n^{\lambda+2}t_2(x) = 0 \) (for \( \mu \geq \pi \)), it follows by the corollary of Lemma 1 that

(8) \( a_n^\mu s(x)t_2(x) = 0 \) for every \( m \geq 2\pi \).

Substituting (5) into (3) and multiplying this equation on the left by \( s(x) \), where \( x = 2x \), we obtain, by (7) and (8),

\[
\lambda a_n^\lambda s(x) + \lambda a_n^\lambda t_2(x) + x\lambda a_n^\lambda s(x)t_1(x) = 0.
\]

The degree of both \( a_n^\lambda s(x) \) and \( a_n^\lambda t_2(x) \) is less than \( n+\nu+1 \), and the coefficient of \( x^{n+\nu+2} \) of \( x\lambda a_n^\lambda s(x)t_1(x) \) is \( a_n^\lambda b_\nu \). Hence \( a_n^\lambda b_\nu = 0 \). But this contradicts (6); hence our assumption that \( a_n^\mu t_2(x) \neq 0 \), for every integer \( \mu \), is false. Thus \( a_n^\mu t_2(x) = 0 \) for some integer \( \lambda \). Now multiplication of (4) on the left by \( a_n^\lambda \) yields \( a_n^\lambda s(x) = 0 \); hence \( a_n^{\lambda+2} = 0 \), q.e.d.


**Lemma 3.** If \( R \) is a PI-ring, then \( R[x] \) is also a PI-ring, and the totalities of the linear-identities of \( R \) and \( R[x] \), respectively, coincide.

The first part of the lemma follows from the fact that \( R \) satisfies a linear identity (Lemma 2 of [4]), and this identity is evidently satisfied by \( R[x] \). If we assume that the operators of \( R \), which are the coefficients of the identities of \( R \), were extended to operate on \( R[x] \) by defining \( \alpha(\sum a_n x^n) = \sum (\alpha a_n) x^n \), the rest of the lemma is readily verified.

The following lemma follows immediately:

**Lemma 4.** A necessary and sufficient condition that a subdirect sum of a set of PI-rings \( \{ Q_a \} \) satisfies an identity \( F(x_1, \cdots, x_m) = 0 \) is that each of the rings \( Q_a \) satisfies the identity \( F = 0 \).

We recall that a PI-ring \( R \) is said to be of degree \( d \) [5] if \( d \) is the minimal degree of the polynomial identities satisfied by \( R \).

**Remark.** It has been shown in [2] that a central simple algebra \( A \) of order \( n^2 \) over its centre is a PI-ring of degree \( 2n \), and the minimal
linear-identity of $A$ is the standard identity $S_{2n}(x) = 0$, uniquely determined up to a numerical factor. Evidently, $A$ satisfies also the identities $S_n(x) = 0$ for every $m \geq 2n$.

We prove now:

**Theorem 1.** If $R$ is a $J$-semi-simple PI-ring of degree $d$, then

1. $d = 2m$.
2. The ring $R$ is a subdirect sum of a set of central simple algebras $\{A_a\}$ such that $m^2$ is the upper bound of the orders of these algebras over their centres.
3. The standard identity $S_d(x) = 0$ is the unique (up to a numerical factor) minimal linear-identity of $R$.

**Proof.** Since $R$ is $J$-semi-simple, $R$ is a subdirect sum of primitive rings $\{A_a\}$ (Theorem 28 of [3]), Lemma 4 implies that each $A_a$ is a PI-ring of degree not greater than $d$. Hence, by Theorem 1 of [4] and by consequence 2 of [5] it follows that each $A_a$ is a central simple algebra of order not greater than $[d/2]$. Let $m^2$ be the upper bound of the orders of the algebras $A_a$; then $m \leq [d/2]$. By the preceding remark it follows that each $A_a$ satisfies the identity $S_{2m}(x) = 0$. Thus, Lemma 4 implies that this identity is satisfied, as well, by their subdirect sum $R$; hence, $d \leq 2m$. On the other hand, $2m \leq 2[d/2] \leq d$. Hence $m = [d/2]$ and $d = 2m$. This completes the proof of the first two parts of the theorem. Since the upper bound $m^2$ is achieved by some $A_a$, and the minimal identities of $R$, whose degree is $2m$, are also identities of this algebra, the proof of the third part of our theorem follows immediately by the preceding remark, that is, by Theorem 7 of [2].

We turn now to the main theorem of this paper:

**Theorem 2.** Let $R$ be an $A$-semi-simple PI-ring of degree $d$, then

1. $d = 2m$.
2. The ring $R$ is a subring of a complete direct sum of central simple algebras $\{A_a\}$ such that $m^2$ is the upper bound of the orders of these algebras over their centres.
3. The identity $S_d(x) = 0$ is the unique (up to a numerical factor) minimal linear-identity of $R$.

**Proof.** Since $R$ is a PI-ring which is $A$-semi-simple, the corollary of Theorem 4 of [5] implies that $R$ is also $K$-semi-simple; hence by Lemma 2 it follows that $R[x]$ is $J$-semi-simple.

In the light of Lemma 3, the application of the preceding theorem to the ring $R[x]$ yields the first and the third parts of the theorem.

\[ \text{Compare with Remark 6 of [2].} \]
The rest of the theorem follows now immediately from the preceding theorem since $R$ is a subring of $R[x]$ which is, by Lemma 3, a PI-ring of degree $d$.

Let $R[x]$ be a subdirect sum of the central simple algebras $\{A_a\}$. By Lemma 4 it follows that the set of the identities satisfied by every $A$ coincides with the set of the identities of the complete direct sum $\sum A_a$ as well as with the totality of the identities of $R[x]$. Hence we obtain, by Lemma 3, the following corollary.

**Corollary 1.** The set of the linear identities of the PI-ring $R$ is the same as the set of the linear identities of the complete direct sum $\sum A_a$.

Let $\{A_a\}$ be a set of central simple algebras of orders not greater than $m^2$. Then each of these algebras satisfies the identity $S_{2m}(x) = 0$. Lemma 3 implies, therefore, that the complete direct sum $\sum A_a$ satisfies the same identity $S_{2m}(x) = 0$. A combination of this fact and the preceding theorem yields:

**Corollary 2.** A necessary and sufficient condition for an A-semi-simple ring to satisfy a polynomial identity is that it be isomorphic to a subring of a complete direct sum of central simple algebras of bounded order.

Another immediate consequence of the preceding theorem is:

**Corollary 3.** Every PI-ring of odd degree contains nonzero nilpotent ideals.

Consider the ring $R$ and the central simple algebras $A_a$ of Theorem 2. Let $F_a$ be a splitting field of the algebra $A_a$. Then $A_a$ is isomorphic with a subring of the total matrix algebra $F_{am}$ of order $m^2$ over $F_a$. The complete direct sum $\sum F_{am}$ of the matrix algebra $\{F_{am}\}$ contains, therefore, a subring isomorphic with the complete direct sum $\sum A_a$. Thus it follows by Theorem 2 that $R$ is isomorphic with a subring of $\sum F_{am}$. It is readily verified that $\sum F_{am}$ is isomorphic with the total matrix ring $F_m$ of order $m^2$ over the complete direct sum $F = \sum F_a$ of the fields $\{F_a\}$. Since $F$ is a direct sum of fields, $F$ is a commutative A-semi-simple ring. Hence, we obtain:

**Theorem 3.** If $R$ is a PI-ring of degree $d$ without nilpotent ideals, then $d = 2m$ and $R$ is isomorphic with a subring of a total matrix ring of order $m^2$ over a commutative ring which does not contain nilpotent ideals.

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*This result has been pointed out to me by the referee.*
Let $R$ be a subring of a total matrix ring of order $m^2$ over a commutative ring. By the proof of [2, Theorem 1] it follows that $R$ is a PI-ring which satisfies the identity $S_{2m}(x) = 0$. Hence, a combination of this fact and the preceding theorem yields:

**Corollary.** An $A$-semi-simple ring $R$ is a PI-ring if and only if $R$ is isomorphic with a subring of a total matrix ring over a commutative ring.

4. **Identities for PI-rings.** Denote by $N = N(R)$ the radical of the PI-ring $R$, that is, the join of all nilpotent ideals of $R$.

In this section we apply the preceding results to obtain identities satisfied by the quotient ring $R/N(R)$.

Let $R$ be a PI-ring of degree $d$, and let $U(R)$ denote the lower radical of $R$. Since $R/U(R)$ is an $A$-semi-simple PI-ring, it follows by Theorem 2, that:

**Theorem 4.** If $R$ is a PI-ring of degree $d$, and $U(R)$ is the lower radical of $R$, then $R/U(R)$ satisfies the identity $S_{2m}(x) = 0$, where $2m \leq d$.

**Theorem 5.** Let $R$ be a PI-ring of degree $d$ such that its radical $N(R)$ is a nilpotent ideal of index not greater than $\rho$, then $S$ satisfies the identity

$$\prod_{i=1}^\rho S(x_{i_1}, \ldots, x_{i_\rho}) = 0. \tag{9}$$

**Proof.** The condition of the theorem implies that $U(R) = N(R)$. Hence, by the preceding theorem, $R/N(R)$ satisfies each of the identities $S(x_{i_1}, \ldots, x_{i_\rho}) = 0$. Since $N(R)\rho = 0$, it is readily seen that $R$ satisfies the identity (9).

By Theorem 2 of [6] it follows that the radical of the quotient ring $R/N(R)$, where $R$ is a PI-ring of degree $d$, is a nilpotent ideal of index not greater than $[d/2]$. Hence we have the following corollary.

**Corollary.** If $R$ is a PI-ring of degree $d$, then $R/N(R)$ satisfies the identity $\prod_{i=1}^{[d/2]} S(x_{i_1}, \ldots, x_{i_\rho}) = 0$.

In a process similar to that of the Laplace expansion of determinants one can readily prove that

$$S_n(x_1, \ldots, x_n) = \sum \pm S_k(x_{i_1}, \ldots, x_{i_k}) S_{n-k}(x_{i_{k+1}}, \ldots, x_{i_n})$$

where the sum ranges over all $C_{n,k}$ different selections of $k$ letters $i_1, \ldots, i_k$ out of $n$ letters, and where $i_{k+1}, \ldots, i_{n}$ denotes the complement of the set $i_1, \ldots, i_k$. This readily implies that the standard

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* Compare with Theorem 9 and its remark of [2].
identity $S_{pq}(x) = 0$ can be expressed as a sum of a set of $q$ products of standard identities each of which is of degree $p$. Hence by the preceding corollary it follows that:

**Theorem 6.** If $R$ is a PI-ring of degree $d$, then $R/N(R)$ satisfies the standard identity $S_p(x) = 0$, where $p = d\lfloor d/2 \rfloor$.

**Bibliography**


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