LINEAR TRANSFORMATIONS ON OR ONTO A BANACH SPACE

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We investigate here a simple property of linear transformations which are not necessarily bounded or closed or one-to-one, but whose domain or range is all of a Banach space.

**Theorem 1.** Let $T$ be a linear transformation from (all of) a Banach space $\mathcal{X}$ onto a normed vector space $Y$. Then there is a number $m > 0$ such that for any $x \in \mathcal{X}$ there exists a sequence $x_n \rightarrow x$ such that $\|Tx_n\| \leq m\|x\|$ and $\{Tx_n\}$ converges in the sense of Cauchy.

**Proof.** Let $C_n$ be the set of all $x \in \mathcal{X}$ such that $\|Tx\| \leq n$ ($n = 1, 2, 3, \ldots$). Then $\mathcal{X} = \bigcup_{n=1}^{\infty} C_n$. In virtue of the Baire category principle there is an integer $k$ such that $C_k$ contains a closed sphere, $S$, whose center and radius we denote by $x_0$ and $r$, respectively. Let $\|Tx_0\| = b$. Thus for each $z$ such that $\|z - x_0\| \leq r$ there exists a sequence $z_n \rightarrow z$ with $\|Tz_n\| \leq k$. Take $m = 2(k + b)/r$.

Now let $x \in \mathcal{X}$ be given. It suffices to consider $x \neq 0$, for if $x = 0$, the theorem is obvious if we use the sequence $x_n = 0$. Let $z = x_0 + rx/\|x\|$. Then $z_n \rightarrow z$ with $\|Tz_n\| \leq k$. Let $x_n = (\|x\|/r)(z_n - x_0)$. Then $x_n \rightarrow x$ and $\|Tx_n\| \leq ((k + b)/r)\|x\| = (m/2)\|x\|$. Now we shall construct a sequence $\{x_n\}$ such that $\{Tx_n\}$ is, in addition, Cauchy convergent. For this we use the following lemma.

**Lemma.** For a given $x \in \mathcal{X}$, $x' \in \mathcal{X}$, there exists a sequence $u_n \rightarrow x$ with $\|Tx' - Tu_n\| \leq (m/2)\|x - x'\|$.

**Proof.** Applying the result already proved to the element $x - x'$ we have $x_n' \rightarrow x - x'$ with $\|Tx_n'\| \leq (m/2)\|x - x'\|$. Let $u_n = x_n' + x_n''$. Then $u_n \rightarrow x$ and $\|Tu_n - Tx\| = \|Tx_n\| \leq (m/2)\|x - x'\|$, as asserted.

To complete the proof of the theorem take $n_1$ large enough so that $\|x - x_n'\| \leq \|x\|/2$ and $\|Tx_n'\| \leq (m/2)\|x\|$. Let $x_1 = x_n'$. By the lemma, $u_n^{(1)} \rightarrow x$ with $\|Tx_1 - Tu_n^{(1)}\| \leq (m/4)\|x\|$. Let $n_2$ be large enough so that $\|u_n^{(1)} - x\| \leq \|x\|/2^2$ and take $x_2 = u_n^{(1)}$. Again by the lemma, there exists $u_n^{(2)} \rightarrow x$ with $\|Tx_2 - Tu_n^{(2)}\| \leq m\|x\|/2^3$. Take $n_3$ large enough so that $\|u_n^{(2)} - x\| \leq (m/2^3)\|x\|$ and let $x_3 = u_n^{(2)}$. Continuing in this manner we have

$$\|Tx\| \leq \frac{m}{2} \|x\|,$$

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Thus \( x_n \to x \) and for \( p > q \geq 1 \):
\[
\| T x_p - T x_q \| = \| T x_p - T x_{p-1} + T x_{p-1} - \cdots + T x_{q+1} - T x_q \| \leq m \| x \| \left( \frac{1}{2^{q+1}} + \cdots + \frac{1}{2^p} \right) \leq m \| x \| / 2^q,
\]
which proves that the \( \{ T x_n \} \) converges in the sense of Cauchy. Finally
\[
\| T x_n \| = \| T x_n - T x_{n-1} + T x_{n-1} - \cdots - T x_1 + T x_1 \| \\
\leq \| T x_n - T x_{n-1} \| + \cdots + \| T x_2 - T x_1 \| + \| T x_1 \| \\
\leq m \| x \| \left( \frac{1}{2^n} + \cdots + \frac{1}{2^q} \right) + \frac{m}{2} \| x \| \leq m \| x \|.
\]

It might be remarked that the closed graph theorem is an immediate corollary of this theorem (that is, if \( T \) is everywhere defined on a Banach space, then it is closed if and only if it is bounded). A further corollary is the fact that if \( T \) is everywhere defined and not closed, then for each \( x \in X \) there exist three sequences \( x_n^{(1)} \to x \), \( x_n^{(2)} \to x \), \( x_n^{(3)} \to x \) with \( T x_n^{(1)} \to y \), \( T x_n^{(2)} \to y \), \( T x_n^{(3)} \to y \) with \( \| y \| \leq m \| x \| \), where \( m \) is independent of \( x \).

**Theorem 2.** Let \( T \) be a linear transformation from a normed vector space \( X \) onto (all of) a Banach space \( Y \). Then there exists a number \( m > 0 \) such that for any \( y \in Y \), there exists a sequence \( y_n \to y \) with \( y_n = T x_n \), \( \| x_n \| \leq m \| y \| \), and \( \{ x_n \} \) convergent in the sense of Cauchy.

**Proof.** The method is entirely analogous to that of Theorem 1 but we give the details. Let \( C_n \) be the set of all \( y \in Y \) such that \( y = T x \) with \( \| x \| \leq n \) \((n = 1, 2, 3, \ldots)\). Then \( Y = \sum_{n=1}^{\infty} C_n \). Hence there exists an integer \( k \) such that \( C_k \) contains a sphere whose center and radius we denote by \( y_0 \) and \( r \) respectively. Say \( y_0 = T x_0 \), with \( \| x_0 \| = b \). Let \( m = 2(b + k)/r \). For any \( z \in Y \) such that \( \| z - y_0 \| \leq r \) there exists \( z_n \to z \) with \( z_n = T \xi_n \) and \( \| \xi_n \| \leq k \). Let \( y \in Y \) be given. Clearly it suffices to consider \( y \neq 0 \). Let \( z = y_0 + (r/\| y \|) \). Then the \( z_n \) described above exists. Let \( y_n' = (\| y \|/r)(z_n - y_0) \). Then \( y_n' \to y \), \( y_n' = T x_n' \) (where \( x_n' = (\| y \|/r)(\xi_n - x_0) \), and \( \| x_n' \| \leq ((k + b)/r) \| y \| = (m/2) \| y \| \).

Now we shall construct a sequence \( \{ y_n \} \) such that \( \{ y_n \} \) is, in
addition, Cauchy convergent. Again we use a lemma.

**Lemma.** For a given \( y \in Y, \ y' = T x' \in Y \) there exists a sequence \( v_n \to y \) with \( v_n = T u_n \) and \( \| u_n - x' \| \leq (m/2) \| y - y' \| \).

**Proof.** Applying the result already established to the element \( y - y' \), we have \( y'' = y - y' \), \( y'' = T x'' \), \( \| x'' \| \leq (m/2) \| y - y' \| \). Set \( v_n = y' + y'' \). Then \( v_n \to y \), \( v_n = T u_n \) (with \( u_n = x' + x'' \)), and \( \| u_n - x' \| = \| x'' \| \leq (m/2) \| y - y' \| \), as asserted. To complete the proof of the theorem select \( n_1 \) large enough so that \( \| y'_{n_1} - y \| \leq \| y \| /2 \). Let \( y'_{n_1} = y_1, \ x'_{n_1} = x_1. \) Then \( y_1 = T x_1, \| x_1 \| \leq (m/2) \| y \|. \) Take \( n_2 \) large enough (by the lemma) so that \( \| v_{n_2} - y \| \leq \| y \| /4, \ v_{n_2} = T u_{n_2}, \) and \( \| u_{n_2} - x_1 \| \leq (m/2) \| y - y_1 \| \leq (m/4) \| y \|. \) Let \( v_{n_2} = y_2, \ u_{n_2} = x_2. \) Take \( n_3 \) large enough so that \( \| v_{n_3} - y \| \leq \| y \| /2^2, \ v_{n_3} = T u_{n_3}, \| u_{n_3} - x_2 \| \leq (m/2) \| y - y_1 \| \leq (m/2^2) \| y \|. \) Let \( v_{n_3} = y_3, \ u_{n_3} = x_3. \) Continuing in this manner we find a sequence \( y_n = T x_n, \| y_n - y \| \leq \| y \| /2^n, \| x_n - x_{n-1} \| \leq (m/2^n) \| y \|. \) Thus \( y_n \to y. \) For \( p > q \geq 1, \)

\[
\| x_p - x_q \| = \left\| x_p - x_{p-1} + x_{p-1} - \cdots + x_{q+1} - x_q \right\| \\
\leq m \| y \| \left( \frac{1}{2^p} + \cdots + \frac{1}{2^{q+1}} \right) \leq \frac{m \| y \|}{2^q}
\]

so that \( \{ x_n \} \) converges in the sense of Cauchy. Finally

\[
\| x_n \| = \left\| x_n - x_{n-1} + x_{n-1} - \cdots + x_2 - x_1 + x_1 \right\| \\
\leq m \| y \| \left( \frac{1}{2^n} + \cdots + \frac{1}{2^2} + \frac{1}{2} \right) \leq m \| y \|.
\]