COMPARISON OF TOPOLOGIES ON FUNCTION SPACES

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1. Introduction. Let $X$ be a topological space, $Y$ be a metrizable space, and $C$ be the collection of continuous functions on $X$ into $Y$. We shall be interested in two topologies on $C$.

The compact-open or $k$-topology. Given compact subset $K$ of $X$ and open subset $W$ of $Y$, denote by $(K, W)$ the collection of functions $f \in C$ such that $f(K) \subseteq W$. The assembly of finite intersections of sets $(K, W)$ forms a base for a topology on $C$, called the compact-open topology, or $k$-topology $^1$. Here the restriction to metrizable spaces $Y$ is unnecessary.

The topology of uniform convergence or $d^*$-topology. If $Y$ is metrizable, let $d$ be a bounded metric consistent with the topology of $Y$. For instance, if $d'$ is an unbounded metric, let

$$d(y_1, y_2) = \min \{d'(y_1, y_2); 1\}, \quad y_1, y_2 \in Y.$$  

Then for any two elements $f$ and $g$ of $C$, define

$$d^*(f, g) = \sup_{x \in X} d(f(x), g(x)) \quad \text{over } X.$$  

It is easily verified that $d^*$ is a metric function, and hence determines a topology on $C$. This topology is called the topology of uniform convergence with respect to $d$, or $d^*$-topology $^3$.

Arens $^1$ and Fox $^2$ have shown that the $k$-topology has certain properties which particularly adapt it to the study of various topological problems, particularly those concerned with homotopy theory. But in case $Y$ is metrizable, so that the $d^*$-topology can be defined, it is obviously easier to deal with than the $k$-topology. Hence it is of interest to inquire when the two topologies are equivalent.

It is easily shown that if $Y$ is metrizable and $X$ is compact, then the $d^*$-topology on $C$ is equivalent to the $k$-topology (see, for instance, $^3$). On the other hand, a theorem of Fox $^2$ implies that if $X$ is a separable metric space which is not locally compact and $Y$ is the real line, then the two topologies are inequivalent. Hu $^3$ gives an example due to Liang Ma wherein $X$ is a countable discrete space, and the two topologies are inequivalent. We shall answer in a fairly general way the question of the relationship between these topologies.

Let $Y$ be metrizable. We shall show in §2, without further restric-

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1 Numbers in brackets refer to the references cited at the end of the paper.
tion on $X$ or $Y$, that the $d^*$-topology is weaker than the $k$-topology (that is, every set open under the $k$-topology is also open under the $d^*$-topology). As remarked above, the compactness of $X$ is a sufficient condition for the equivalence of the two topologies. We shall show in §3 that if $X$ is completely regular and $Y$ contains a nondegenerate arc, then the compactness of $X$ is also a necessary condition for the equivalence of the $d^*$-topology and the $k$-topology.

2. Theorem. If $Y$ is metrizable, then the $d^*$-topology is weaker than the $k$-topology.

We must show that if $U$ is open under the $k$-topology and $f \in U$, then we can find $\varepsilon > 0$ such that if $d^*(f, g) < \varepsilon$, then $g \in U$.

By hypothesis, there exist compact subsets of $X$, $K_1, \ldots, K_n$, and open subsets of $Y$, $W_1, \ldots, W_n$, such that

$$f \in (K_1, W_1) \cap \cdots \cap (K_n, W_n) \subset U.$$ 

Since each $K_i$ is compact, so is each $f(K_i)$. Also, each $f(K_i)$ is disjoint from the corresponding closed set $Y - W_i$. But the distance between two disjoint subsets of metric space, one of which is compact and the other closed, is positive. Let $\varepsilon$ be a positive number less than the smallest of the distances from the $f(K_i)$ to the corresponding $Y - W_i$.

Now suppose $d^*(f, g) < \varepsilon$. If $x \in K_i$, then the distance from $g(x)$ to $f(K_i)$ is less than $\varepsilon$, so $g(x) \in W_i$. Thus

$$g \in (K_1, W_1) \cap \cdots \cap (K_n, W_n) \subset U,$$

so $g \in U$, as required.

3. Theorem. Let $X$ be a completely regular space, and let $Y$ be a metrizable space containing a nondegenerate arc. Then a necessary condition that the $d^*$-topology and the $k$-topology be equivalent is that $X$ be compact.

Let $\phi: I \to Y$ define a nondegenerate arc, where $I$ denotes the closed unit interval. Pick $t_i \in I$ so $y_1 = \phi(t_i)$ is different from $y_0 = \phi(0)$. Choose $\varepsilon > 0$ so $d(y_0, y_1) \geq \varepsilon$. Define $f$ as the function of $C$ which is constantly equal to $y_0$.

Suppose the two topologies are equivalent. Then there must be compact subsets of $X$, $K_1, \ldots, K_n$, and open subsets of $Y$, $W_1, \ldots, W_n$, such that

$$f \in (K_1, W_1) \cap \cdots \cap (K_n, W_n) \subset S,$$

where $S$ is the set of functions $g \in C$ such that $d^*(f, g) < \varepsilon$.

We complete the proof by contradiction. If $X - (K_1 \cup \cdots \cup K_n)$
is empty, then $X$ is the union of a finite number of compact sets, and hence must be compact.

Otherwise, choose $x_0 \in X - (K_1 \cup \cdots \cup K_n)$. Since $x_0$ and $(K_1 \cup \cdots \cup K_n)$ are disjoint closed subsets of the completely regular space $X$, there exists a continuous function $\theta: X \to I$ with $\theta(x) = 0$ on $(K_1 \cup \cdots \cup K_n)$ and $\theta(x_0) = t_1$. Define $g: X \to Y$ as the product $\phi \theta$; $g(x) = \phi(\theta(x))$. It is obvious that $g \in C$. Moreover, for $x \in K_i$, we have that $g(x) = \phi(\theta(x)) = \phi(0) = y_0 \in W_i$; so $g(K_i) \subset W_i$, and $g \in (K_i, W_i) \cap \cdots \cap (K_n, W_n)$.

On the other hand,

$$d^*(f, g) \geq d(f(x_0), g(x_0)) = d(y_0, \phi(\theta(x_0)))$$

$$= d(y_0, \phi(t_1))$$

$$= d(y_0, y_1) \geq \varepsilon.$$ 

Hence $g \in S$. This contradiction completes the proof.

4. A generalization. If instead of requiring $Y$ to be metrizable, we insist only that it be a uniform space, a topology of uniform convergence can be defined on $C$. In this case the compactness of $X$ is a sufficient condition for the equivalence of the topology of uniform convergence and the $k$-topology, the obvious analogue of Theorem 2 holds without further restriction, and the analogue of Theorem 3 holds if we require that $Y$ have a separated uniform structure. The proofs are not essentially different from those above, but are somewhat longer.

References