CONTINUITY OF AREA FOR HARMONIC SURFACES WITH BOUNDARIES OF UNIFORMLY BOUNDED LENGTH

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The object of this note is to supplement a paper by Young [1], On the isoperimetric ratio for a harmonic surface, by answering in the affirmative a question raised by Morse and Tompkins [2]. Continuity of area as the harmonic surface $S^*$ shrinks to a point is dealt with by Young by means of his main inequality. Here the same machinery deals with the more general case in which $S^*$ approximates another surface $S$.

In what follows $r = |u + iv|$, $D$ is the disc $r \leq 1$, $R$ its rim, and, in correspondence with a real number $t$ where $0 < t < 1$, $D^t$ is the concentric disc $r \leq 1 - t$ and $R^t$ is the annulus $1 - t \leq r \leq 1$. We shall divide $R^t$ into $n$ parts $\lambda$ in which it is met by the lines amp $(u + iv) = 2\pi k/n$, where $-n/2 < k \leq n/2$. The suffixes $t$, or $k$, $n$, $t$ in $D^t$ and $R^t$, or in $\sigma$ and $\lambda$, are understood to have been dropped.

$S$ and $S^*$ denote continuous vector functions $x(u, v)$ and $x^*(u, v)$ on $D$ and it is assumed that $x^*(u, v)$ is the harmonic extension to $D$ of a continuous vector function on $R$. We shall assume further that $(1 - r)|x_u + ix_v|$ is bounded in $D$, and we denote by $b$ its supremum; $b^*$ is similarly defined in terms of $S^*$ [and is finite since $x^*(u, v)$ is harmonic].

We write $d(S, S^*)$ and $\rho(S, S^*)$ respectively for the suprema in $D$ of the expressions $|x(u, v) - x^*(u, v)|$ and $(1 - r)|x_u - x^*_u + i(x_v - x^*_v)|$. Moreover, given any subset $E$ of $D$ whose boundary is a simple closed curve $C$, we write $\delta(E)$ for the diameter of the set of values $x(E)$ taken by $x(u, v)$ in $E$, and $A(E), L(E)$ respectively for the area and boundary-length defined by the expressions

$$\int \int_E \left\{ x_u^2 + x_v^2 - (x_u^* + x_v^*)^2 \right\}^{1/2} dudv, \quad \int_C |dx| ;$$

$\delta^*(E), A^*(E), L^*(E)$ denote the similar quantities for $S^*$ and the argument $E$ is omitted when $E = D$.

**Lemma 1.** For the sum of the angles $\alpha$ subtended at $\cos \gamma + isin \gamma$ by the $n$ segments $\lambda$, we have the estimate $\sum_{\alpha < \pi}$ provided that $t \leq \eta(n)$ where $\eta(n) = \left[ n(1 + \log n) \right]^{-1}$.

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Numbers in brackets refer to the references at the end of the paper.
Proof. We first estimate the angle $\alpha$, subtended by the single segment $\lambda = \overline{AB}$ on which $\text{amp} (u + iv) = \theta$, in terms of $\beta = |\theta - \gamma| / 2$ where $\theta - \gamma$ is reduced mod $2\pi$ so that $2\beta \leq \pi$. The diameter $AC$ containing $AB$ subtends a right angle at $M = \cos \gamma + i \sin \gamma$ and we have two expressions for the sine of the angle $\phi$ subtended by $MC$ at $B$:

$$2 \sin \beta \sin \alpha / t = \frac{AM}{MC} \sin \alpha / AB = \sin \phi$$

$$= \frac{MC \cos \alpha / BC}{2 \cos \beta \cos \alpha / (2 - t)}.$$

Hence if $\beta \neq 0$ we find that $\alpha < t \cot \beta / (2 - t) < t \cot \beta < t / \beta$; moreover, in any case, $\alpha \leq \pi / 2$.

To establish our lemma, in which we may suppose by circular symmetry that $0 \leq \gamma \leq \pi / n$, we have, on denoting by $\sum'$ and $\sum''$ sums for $-n / 2 < k < 0$ and $0 < k \leq n / 2$,

$$\sum \alpha < \frac{\pi}{2} + (\sum' + \sum'') \frac{t}{\beta} = \frac{\pi}{2} + t \sum' \left( \frac{\gamma}{2} - \frac{k \pi}{n} \right)^{-1}$$

$$+ t \sum'' \left( \frac{k \pi}{n} - \frac{\gamma}{2} \right)^{-1}$$

$$\leq \frac{\pi}{2} + t \sum' \left( - \frac{k \pi}{n} \right)^{-1} + t \sum'' \left[ \left( k - \frac{1}{2} \right) \frac{\pi}{n} \right]^{-1}$$

$$\leq \frac{\pi}{2} + \frac{2nt}{\pi} \left\{ 1 + \frac{1}{2} + \cdots + \frac{1}{n - 1} \right\}$$

$$< \frac{\pi}{2} + \frac{2t}{\pi} n(1 + \log n) < \frac{\pi}{2} \left[ 1 + \ln(1 + \log n) \right] \leq \pi.$$

**Lemma 2.** For every $S^*$ the inequality $t \leq \eta(n)$ implies $\sum L^*(\sigma) \leq 5L^*.$

**Proof.** Let $\Lambda = \sum \int_A |dx^*|$. By Young's variant [1, p. 400, (3.2)] of F. Carlson's inequality [3], we have, since $x^*$ is harmonic,

$$\Lambda \leq \pi^{-1} \int_R \left( \sum \alpha \right) |dx^*| < \int_R |dx^*| = L^*,$$

provided that $t \leq \eta(n)$. Our assertion now follows from the relation $\sum L^*(\sigma) = 2\Lambda + L^* + L^*(D^-)$ where the last term on the right cannot exceed $2L^*$ (by [1, p. 405 middle of page]; actually $L^*(D^-) \leq L^*$).

**Lemma 3.** There is an absolute constant $K$ and, given $S$ and $\epsilon > 0$, a number $t = t_0(\epsilon, S) < \epsilon$, such that the inequality $d(S, S^*) < \epsilon$ implies $A^*(R^+) \leq \epsilon KL^*.$

**Proof.** By uniform continuity of $x(u, v)$, there exist $n = n(\epsilon, S)$ and $t = t_0(\epsilon, S) \leq \eta(n)$, so that $\delta(\sigma) < \epsilon$. This evidently insures $\delta^*(\sigma)$
<3\varepsilon$, while by Lemma 2, $\sum L^*(\sigma) \leq 5L^*$. Consequently (applying Young’s isoperimetric inequality $A \leq KLD$ of [1] to the harmonic surface $x^*(u, v)$ on $\sigma$ and summing),

$$A^*(R^+) = \sum A^*(\sigma) \leq K \sum L^*(\sigma) \delta^*(\sigma) \leq 15\varepsilon KL^*,$$

which is of the asserted form, with $15\varepsilon$ for $K$.

**Lemma 4.** There is an absolute constant $K$ so that the inequality $\rho(S, S^*) < \varepsilon$ implies $|A(D^-) - A^*(D^-)| \leq K\varepsilon(b + b^*)^{3/2}$.

**Proof.** Since the quantities $|x_u + ix\|/\varepsilon$, $|x^* + ix^*|/\varepsilon$, and $|(x_u - x^*_u) + i(x_u - x^*_u)|/\rho(S, S^*)$ do not exceed $(1 - r)^{-1}$, the expression $U = (x^*_u - x^*_u)^2$ differs from the corresponding expression $U^*$ by at most $K^2(1 - r)^{-4}\rho(S, S^*)(b + b^*)^3$ where $K$ is an absolute constant. Since

$$|U^{1/2} - U^*^{1/2}| \leq \left\{ \frac{|U^{1/2} - U^*^{1/2}|}{(U^{1/2} + U^*^{1/2})^{1/2}} \right\}^{1/2} = |U - U^*|^{1/2},$$

it follows that

$$|A(D^-) - A^*(D^-)| \leq \int \int_{D^-} |U - U^*|^{1/2} dudv$$

$$= 2\pi K\varepsilon(b + b^*)^{3/2} \int_{0}^{1} (1 - r)^{-2} r dr$$

which implies our assertion, because

$$\int (1 - r)^{-2} r dr \leq \int (1 - r)^{-2} r dr = t^{-1} - 1 < t^{-1}.$$

**Lemma 5.** Given $S$ and the constant $N$ and given $\varepsilon > 0$, there exists $\varepsilon'' = \varepsilon''(S, N, \varepsilon') > 0$ such that the relations $\rho(S, S^*) + d(S, S^*) < \varepsilon''$, $L^* \leq N$, $A < \infty$ together imply $|A - A^*| < \varepsilon'$ for every harmonic $S^*$.

**Proof.** Clearly $\delta^* \leq L^*$ and by an inequality of Schwarz [1, p. 398, (2.3)] $b^* \leq 2b^*$. Moreover $b \leq b^* + \rho(S, S^*)$, so that $b < \infty$. We may therefore suppose $N \geq b$ without loss of generality (since $b$ depends only on $S$) and we then have $b + b^* \leq 3N$.

Given $\varepsilon > 0$, we now determine $t = t(\varepsilon, S) < t_0(\varepsilon, S)$, where $t_0(\varepsilon, S)$ is defined as in Lemma 3, so that

$$A(R^+) < \varepsilon.$$
This is possible since $A$ is finite. Then the inequality $\rho(S, S^*) + d(S, S^*) < t^4$ implies on one hand by Lemma 3 (since $t^4 < t < \varepsilon$)

$$A^*(R^+) \leq KN\varepsilon,$$

and on the other hand by Lemma 4

$$|A(D^-) - A^*(D^-)| < K\varepsilon(3N)^{3/2}.$$

Hence by addition the quantity

$$|A - A^*| = |A(R^+) - A^*(R^+) + \{A(D^-) - A^*(D^-)\}|$$

is less than $\varepsilon + KN\varepsilon + K\varepsilon(3N)^{3/2} = \varepsilon'$ provided our hypotheses are satisfied and $\varepsilon'' = t^4$ where $t = t(\varepsilon, S)$. This completes the proof.

**Continuity Theorem**. If $S$ is harmonic and $N$ fixed, then for harmonic $S^*$ the relations $d(S, S^*) \to 0$ and $L^* \leq N$ imply $A^* \to A$.

**Proof.** By semi-continuity of area we may suppose $A < \infty$. It is now sufficient to apply Lemma 5 with $3\varepsilon''$ in place of $\varepsilon''$ since $\rho(S, S^*) \leq 2d(S, S^*)$ by the inequality of Schwarz already referred to.

The theorem just proved answers the question raised by Morse and Tompkins [2, p. 828]. We do not know, however, whether the result remains true if $d(S, S^*)$ is replaced by the Fréchet distance of the harmonic surfaces $S$ and $S^*$.

**References**

3. F. Carlson, Arkiv för Matematik, Astronomi och Fysik vol. 29B (1943) no. 11.

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