A THEOREM ON DIMENSION

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Every \( n \)-dimensional separable metric space can be homeomorphically imbedded in the closed \((2n+1)\)-cube \( I^{2n+1} \). This, of course, does not characterize \( n \)-dimensionality, for spaces of dimension greater than \( n \) can be imbedded homeomorphically in \( I^{2n+1} \). A theorem due to Hurewicz\(^1\) suggests that the existence of a more general kind of mapping into the \( n \)-cube \( I^n \) may characterize \( n \)-dimensionality for compact spaces. For compact spaces this theorem reduces to: If \( X, Y \) are compact and separable metric, \( f:X \to Y \) is continuous, and \( \dim X \geq \dim Y \), then there exists \( y \in Y \) such that \( \dim f^{-1}(y) \geq \dim X - \dim Y \). This suggests the possibility that if \( \dim X = \dim Y \), one may be able to modify the map \( f \) slightly to obtain a map \( g \), so that each \( g^{-1}(y) \) has dimension zero, or is light. While this is not true in general, we show that it is possible if \( \dim X = n \) and \( Y = I^n \).

Lemma 1. Let \( P^n \) be an \( n \)-polyhedron and \( I^n \) be the \( n \)-cube, both with given simplicial decompositions. If \( \psi:P^n \to I^n \) is simplicial and \( \delta > 0 \), then there exists a map \( \phi:P^n \to I^n \) such that:

1. For any \( x \in P^n \), \( \|\phi(x) - \psi(x)\| < \delta \).
2. For any \( y \in I^n \), \( \phi^{-1}(y) \) has at most one point in each (open) simplex of \( P^n \).

Proof. Let \( 0 < \eta < \min (\delta/2, \xi/2) \) where \( \xi \) is the minimum diameter of the collection of \( n \)-simplexes of \( I^n \). For each vertex \( q \) in the decomposition of \( I^n \) let \( S^{n-1}(q) = \{ y \in I^n , \| y - q \| = \eta \} \). On each \( S^{n-1}(q) \) select \( N_q \) distinct points such that if \( q \) and \( q' \) are two vertices with \( q' \in \) the closure of the set \( St(q) \), then no \((n-1)\)-hyperplane determined by \( n \) of the selected points of \( S^{n-1}(q) \) coincides with an \((n-1)\)-hyperplane determined by \( n \) of the selected points of \( S^{n-1}(q') \).

Let \( \{ p_i \} \) denote the collection of vertices of \( P^n \). If \( \psi^{-1}(q) = \bigcup P_i \), then let \( \{ \phi'(p_i) \} \) be distinct points from the collection defined on \( S^{n-1}(q) \). This defines \( \phi' \) on all the vertices of \( P^n \), and, if \( \sigma_i \) is an \( l \)-simplex, then \( \phi' \) sends its vertices into \( l+1 \) independent points of \( I^n \). Extend \( \phi' \) linearly on each simplex of \( P^n \) to obtain a map \( \phi:P^n \to I^n \). It can be seen that \( \phi \) has the required properties.

Definition. A map \( f:X \to Y \) is said to be \( \epsilon \)-light provided that:
For any \( y \in Y \), each component of \( f^{-1}(y) \) has dia \(< \epsilon \).

\(^1\)Dimension theory, Hurewicz and Wallman, Princeton, 1948, Theorem VI, 7, p. 91.

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Given two spaces $X, Y$ and a sequence $\{\epsilon_i\} \to 0$. Let $F_{\epsilon_i}$ denote the set of all $\epsilon_i$-light maps of $X$ into $Y$. Then $\bigcap_i F_{\epsilon_i}$ is all light maps $X \to Y$.

**Theorem.** Let $X$ be a compact metric space. Then $\dim X \leq n$ if and only if there exists a light map of $X$ into $I^n$.

**Proof.** Necessity. Let $F$ denote the function space of maps $f : X \to I^n$ with the uniform topology, and denote by $F_{\epsilon_i}$ the set of $\epsilon_i$-light elements of $F$.

(a) For any $\epsilon > 0$, $F_{\epsilon_i}$ is open in $F$.

Suppose that $f \in F_{\epsilon_i}$ and $\{h_i\} \to f$ with each $h_i \in F - F_{\epsilon_i}$. Then for each $i$ there is a point $y_i \in I^n$ and a component $c_i$ of $h_i^{-1}(y_i)$ such that $\text{diam} c_i \leq \epsilon$. Let $y$ be a limit point of $\{y_i\}$. Then $\{c_i\}$ contains a subsequence converging to a closed connected set $C$ with $\text{diam} c \leq \epsilon$. Furthermore $C \subseteq f^{-1}(y)$, contradicting the assumption that $f \in F_{\epsilon_i}$.

(b) For any $\epsilon > 0$, $F_{\epsilon_i}$ is dense in $F$.

Let $h \in F$ and $\delta > 0$. We must find $f \in F_{\epsilon_i}$ such that for any $x \in X$, $\|f(x) - h(x)\| < \delta$.

Take a simplicial decomposition of $I^n$ into simplexes of $\text{diam} < \delta/2$. If $\{q_i\}$ are the vertices, then $\{h^{-1}(\text{St}(q_i))\}$ is a covering of $X$ by open sets. Take a covering $V$ of $X$ such that

1. $V$ is a refinement of $\{h^{-1}(\text{St}(q_i))\}$,
2. Each $v \in V$ has $\text{diam} < \epsilon$,
3. Order $V \leq n + 1$.

Denote by $N(V)$ the nerve of $V$ and by $\xi$ a barycentric $V$-mapping $\xi : X \to N(V)$. Let $\{p_i\}$ be the collection of vertices of $N(V)$. Then the rule $\psi'(p_i) = \text{some } q_i$ such that $h(q_i) \subseteq \text{St}(q_i)$ defines a simplicial map $\psi' : N(V) \to I^n$. By the lemma we can find a map $\psi : N(V) \to I^n$ satisfying conditions (1) (with $\delta/2$) and (2). The combined map $f = \psi \xi$ has the property that $\|f(x) - h(x)\| < \delta$ for every $x \in X$; furthermore $f \in F_{\epsilon_i}$. For let $C$ be a connected set in $X$ such that $f(C)$ is a single point $y \in I^n$. Then $\xi(C)$ is a connected set in $N(V)$, but it must be contained $\psi^{-1}(y)$ which is totally disconnected. Thus $\xi(C)$ is a single point, and, since $\xi$ is a barycentric $V$-mapping, this implies $\text{diam } C < \epsilon$.

(y) The function space $F$ is complete so that we may apply a theorem of Baire stating that the intersection of a countable number of open dense sets of a complete space has an intersection which is dense in the space. Thus if $\{\epsilon_i\} \to 0$, then the collection of light maps of $X$ into $I^n(= \bigcap_i F_{\epsilon_i})$ is dense in $F$. This completes the proof of the necessity.

Sufficiency. This follows from the theorem of Hurewicz mentioned in the introduction.

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Remark 1. The condition in the theorem that $X$ be compact is important. For if $X$ is the set of points in the plane which have at least one coordinate a rational number, then $\dim X = 1$, but there is no light closed map of $X$ into the real line, and, a fortiori, none into the unit interval.

Remark 2. It is clear that $I^n$ could be replaced in the theorem by any $n$-manifold. This contrasts with the imbedding theorem, where there are $n$-dimensional sets that can be imbedded in manifolds of dimension less than $2n+1$, but cannot be imbedded in $I^{2n}$.

Added in proof. Our attention has been called to a paper by M. Katětov, On rings of continuous functions, Časopis pro Pěstování Matematiky a Fysiky vol. 75 (1950) pp. 1–16, Mathematical Reviews vol. 12 (1951) p. 119, which apparently contains several theorems closely related to our result, though not the same.