ON AN EQUIVALENT DEFINITION OF THE TRANSFINITE DIAMETER

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Let
\[ f(z) = a_0 + a_1/z + \cdots, \quad a > 0, \]
map the exterior of the unit circle \( C \) onto the exterior of a simply-connected domain \( B \). It is part of a general theory (see e.g. [3]) that the quantity \( 1/a \) can be identified with the transfinite diameter of \( B \). In particular, let \( T_n(z) \) be the polynomial of degree \( n \) with leading coefficient 1 whose \( L^\infty \) norm over \( B \), i.e. the maximum of whose absolute value over \( B \), is minimum. Then
\[ \lim_{n \to \infty} (M_n)^{1/n} = a. \]

The object of this paper is to show that the last result holds true in the case of domains with a Jordan boundary if we replace the minimal polynomials in the \( L^\infty \) metric by those in any \( L^p \) metric, \( p \geq 2 \). Let \( Q_n^{(p)}(z) \) be the polynomial minimizing
\[ \left( \int_B |Q_n(z)|^p dxdy \right)^{1/p} \]
among all polynomials of degree \( n \) with leading coefficient 1, and let \( \lambda_n^{(p)} \) be the \( L^p \) norm of \( Q_n^{(p)}(z) \), i.e. the value of (2) for \( Q_n = Q_n^{(p)} \). Then \( \lim_{n \to \infty} (\lambda_n^{(p)})^{1/n} = a \). After a preliminary lemma substantially due to Carleman [2], we shall prove the result first for \( p = 2 \), and then for \( 2 < p < \infty \).

LEMMA. Let \( B \) have an analytic boundary. Then \( \lim_{n \to \infty} (\lambda_n^{(p)})^{1/n} = a \).

PROOF. Since \( T_n(z) \) is a competing polynomial in the \( L^2 \) minimum problem,
\[ (\lambda_n^{(2)})^2 \leq \int_B |T_n(z)|^2 dxdy \leq M_n^2 \cdot A, \]
where \( A \) is the area of \( B \), and
\[ \lim_{n \to \infty} sup (\lambda_n^{(2)})^{1/n} \leq \lim_{n \to \infty} (M_n)^{1/n} = a. \]

On the other hand, since \( B \) has an analytic boundary, the function

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\[ f(\xi) \text{ can be continued to be analytic and schlicht up to a circle } \gamma_{\xi}; \quad |\xi| = \rho_1 < 1. \] Let \( \rho_1 < \rho < 1 \) and let \( D \) be the annulus bounded by \( \gamma_{\rho} \) and \( \gamma_1 \); furthermore, let \( f(D) \) be its image in \( B \). Then

\[
\int \int_B |Q_n^{(2)}(z)|^2 \, dx \, dy \geq \int \int_{f(D)} |Q_n^{(2)}(z)|^2 \, dx \, dy
\]

(5)

\[
= \int \int_D |Q_n^{(2)}[f(\xi)]f'(\xi)|^2 \, d\xi \, d\eta.
\]

Now \( Q_n^{(2)}(\xi) = \zeta^n + \cdots + \), \( f(\xi) = a_1 + a_0 + a_1/\xi + \cdots \); hence

\[ Q_n^{(2)}[f(\xi)]f'(\xi) = a_0 + 1 + \sum_{j=1}^{\infty} b_j \xi^{-j} \]

and

\[
(\lambda_n^{(2)})^2 = \int \int_D \left| a_0 + 1 + \sum_{j=1}^{\infty} b_j \xi^{-j} \right|^2 \, d\xi \, d\eta
\]

(6)

\[
= 2\pi a^{2n+2} \int_0^1 r^{2n+1} \, dr + 2\pi \sum_{j=1}^{\infty} b_j^2 \int_0^1 r^{2n-2j+1} \, dr
\]

\[
= \frac{\pi a^{2n+2}}{n+1} (1 - \rho^{2n+2}).
\]

Since \( \rho < 1 \), it follows that \( \rho^{2n+2} \to 0 \), and \( \liminf_{n \to \infty} (\lambda_n^{(2)})^2 \geq a \). By comparing (4) and (6) we see that \( \lim_{n \to \infty} (\lambda_n^{(2)})^2/n \) exists and equals \( a \).

**Theorem 1.** Let \( B \) be a simply-connected domain with a Jordan boundary. Then \( \lim_{n \to \infty} (\lambda_n^{(2)})^{1/n} \) exists and equals \( a \).

**Proof.** Let \( B \supset B' \). Then if \( Q_n^{(2)}(z) \) is the \( L^2 \) minimal polynomial over \( B \),

\[
[\lambda_n^{(2)}(B)]^2 = \int \int_B |Q_n(z)|^2 \, dx \, dy \geq \int \int_{B'} |Q_n(z)|^2 \, dx \, dy \geq [\lambda_n^{(2)}(B')]^2.
\]

Hence

\[
\limsup_{n \to \infty} [\lambda_n^{(2)}(B)]^{1/n} \leq \limsup_{n \to \infty} [\lambda_n^{(2)}(B')]^{1/n};
\]

the same is true for \( \liminf \) and \( \lim \) if it exists. Thus, we see that \( \limsup_{n \to \infty} [\lambda_n^{(2)}(B)]^{1/n} \) and \( \liminf_{n \to \infty} [\lambda_n^{(2)}(B)]^{1/n} \) are increasing set functions. Now an arbitrary domain can be approximated from the exterior by domains with analytic boundaries in such a way that the
respective $a(B)$'s are arbitrarily close to each other; one needs only to take level lines of the exterior mapping function. In the case of a domain with a Jordan boundary, it follows from the Carathéodory theory [1] that this is also true for interior approximation. Since the transfinite diameter is also an increasing set function, it follows by approximation that $\lim (\lambda_n^{(p)})^{1/n}$ exists and equals $a$ in the case of an arbitrary Jordan domain.

**Theorem 2.** Let $Q_n^{(p)}(z)$ be the polynomial of degree $n$ with leading coefficient 1 minimizing

$$\left( \int \int_B |Q_n(z)|^p dxdy \right)^{1/p}$$

and let $\lambda_n^{(p)}$ be the corresponding minimum value of (9). Then for any $p > 2$, $\lim_{n \to \infty} (\lambda_n^{(p)})^{1/n}$ exists and equals $a$.

**Proof.** For any $f(z) \in L^2$ and $L^p$, $p > 2$, we have

$$\left( \int \int_B |f(z)|^2 dxdy \right)^{1/2} \leq \left( \int \int_B |Q_n^{(p)}(z)|^p dxdy \right)^{1/p}.$$ 

Therefore

$$\lambda_n^{(2)} \leq \left( \int \int_B |Q_n^{(p)}(z)|^2 dxdy \right)^{1/2} \leq \left( \int \int_B |Q_n^{(p)}(z)|^p dxdy \right)^{1/p} = \lambda_n^{(p)}$$

and hence

$$a = \lim_{n \to \infty} (\lambda_n^{(2)})^{1/n} \leq \liminf_{n \to \infty} (\lambda_n^{(p)})^{1/n}.$$ 

On the other hand,

$$\lambda_n^{(p)} \leq \int \int_B \left| T_n(z) \right|^p dxdy \leq A [M_n]^p,$$

where $A$ is the area of $B$. Hence

$$\limsup_{n \to \infty} (\lambda_n^{(p)})^{1/n} \leq \lim (M_n)^{1/n} = a.$$ 

Combining (11) and (12), we obtain that $\lim_{n \to \infty} (\lambda_n^{(p)})^{1/n}$ exists and equals $a$.

**Bibliography**

1. C. Carathéodory, *Untersuchungen ueber die konformen Abbildungen von festen*


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