COMPLEX STRUCTURES ON REAL BANACH SPACES

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1. Let $E_0$ be a topological vector space over the complex number field $\mathbb{C}$. The mapping $(\xi, x) \rightarrow \xi x$ is then continuous when one restricts $\xi$ to take only real values. Hence, we have on $E_0$ a structure of topological vector space over the real number field $\mathbb{R}$. We shall denote by $E$ that topological vector space. The homothetic mapping $x \rightarrow ix$ of $E_0$ onto itself is an automorphism $u$ of the topological vector space $E$, such that $u^2(x) = -x$. Conversely, let $E$ be a topological vector space over $\mathbb{R}$, and let $u$ be an automorphism of $E$ such that $u^2(x) = -x$. One can then define on $E$ a structure of vector space over $\mathbb{C}$, by setting $(\lambda + i\mu)x = \lambda x + \mu u(x)$. The axioms of vector spaces are trivially verified, and the continuity of $u$ insures that the mapping $(\xi, x) \rightarrow \xi x$ of $\mathbb{C} \times E$ into $E$ is continuous. One defines thus a topological vector space $E_0$ over $\mathbb{C}$, from which the original space $E$ can be derived as above.

When a topological vector space $E$ over $\mathbb{R}$ is given, the question naturally arises of the existence of an automorphism $u$ of $E$ such that $u^2(x) = -x$. It is well known that when $E$ has finite dimension $n$, the necessary and sufficient condition for the existence of $u$ is that $n$ be an even number. In this note, we shall give an example of an infinite-dimensional Banach space $E$ over $\mathbb{R}$, such that there exists no automorphism $u$ of $E$ with the required property.

2. When $E$ is a Banach space over $\mathbb{R}$, and an automorphism $u$ of $E$ such that $u^2(x) = -x$ exists, the topology of the space $E_0$ (which is identical with the topology of $E$) can still be defined by a norm, for instance $\|x\|_0 = \sup_{\xi \in \mathbb{C}} \|e^{i\xi}x\|$ meaning the norm on $E$; for one has obviously $\|x\| \leq \|x\|_0 \leq (1 + \|u\|)\|x\|$. Let $E'$ and $E_0'$ be the dual spaces of $E$ and $E_0$ respectively. $E'$ is a Banach space over $\mathbb{R}$ and $E_0'$ a Banach space over $\mathbb{C}$. There is a well known natural mapping of $E'$ onto $E_0'$ which to every continuous linear form $v \in E'$ associates the continuous linear form $w = \phi(v)$ over $E_0$ such that $w(x) = v(x) - iv(ix)$. The inverse mapping $v = \psi(w)$ is such that $v(x) = \Re(w(x))$. As $\|w(x)\| \leq (1 + \|u\|)\|v\|\|x\|$, and $\|v(x)\| \leq \|w(x)\|$, it is clear that $\phi$ and $\psi$ are continuous; $\phi$ is therefore an isomorphism of the topological vector space $E'$ over $\mathbb{R}$ onto the topological vector space $E_0'$ over $\mathbb{R}$.

Now let $E''$ be the dual of the Banach space $E'$ (over $\mathbb{R}$). Let $E_0''$
be the dual of the Banach space $E'_0$, when $E'_0$ is considered as a space over $\mathbb{R}$; and let $E''_0$ be the dual of the Banach space $E'_0$, when $E'_0$ is considered as a space over $\mathbb{C}$. The same argument as before yields a natural mapping $\phi'$ of $E''_0$ onto $E''_0$, which is an isomorphism for the structures of topological vector spaces over $\mathbb{R}$ of these spaces. Moreover, the mapping which to any continuous linear form $V$ over $E'$ associates the continuous linear form $w \mapsto V(\psi(w))$ over $E''_0$ (considered as a space over $\mathbb{R}$) is again an isomorphism of $E''_0$ onto $E''_0$ (for the structures of topological spaces over $\mathbb{R}$). We thus get finally a natural isomorphism $\Phi$ of $E''_0$ onto $E''_0$, when both these spaces are considered as topological vector spaces over $\mathbb{R}$.

Moreover, there is a natural isomorphism $x \mapsto U_x$ of the Banach space $E$ into the Banach space $E''_0$, such that $U_x(v) = v(x)$ for every $v \in E'$. Similarly, there is a natural isomorphism $x \mapsto U_x$ of the Banach space $E_0$ (over $\mathbb{C}$) into the Banach space $E''_0$ (over $\mathbb{C}$). For every $x \in E_0$, and every $w \in E'_0$, one has $U_x(\psi(w)) = \Re(w(x))$, and therefore, if $T(w) = U_x(\psi(w))$, and $W = \phi'(T)$, one has $W(w) = \Re(w(x)) - i\Im(w(ix)) = w(x) = U_0(w)$. In other words, $\Phi(U_x) = U_0$ for every $x \in E$, which means that under the isomorphism $\Phi$, $E$ (considered as imbedded in $E''_0$) is transformed into $E_0$ (considered as imbedded in $E''_0$).

3. Now R. C. James has given recently an example of a Banach space $E$ over $\mathbb{R}$, such that $E''_0/E$ has dimension one over $\mathbb{R}$. Suppose there existed an automorphism $u$ of $E$ such that $u(x) = -x$; it would define a Banach space $E_0$ over $\mathbb{C}$, and it follows from §2 that $E''_0/E_0$ would have dimension one over $\mathbb{R}$. But $E''_0/E_0$ is a vector space over $\mathbb{C}$ which is not reduced to 0, and as such its dimension over $\mathbb{R}$ is at least 2. We thus reach a contradiction, which proves our contention.

The same example exhibits another interesting feature concerning the problem we are considering. Namely, there exists an increasing sequence $(L_n)$ of closed subspaces of $E$, whose union $M$ is dense in $E$, and a (noncontinuous) one-to-one linear mapping $u$ of $M$ onto itself, such that $u^2(x) = -x$, and that $u$, restricted to any one of the $L_n$, is a (continuous) automorphism of that subspace. In fact, it is known that there exists in $E$ two closed subspaces $H_1$, $H_2$, each of which is (as a topological vector space) isomorphic to a separable Hilbert space, and such that $H_1 \cap H_2 = \{0\}$, and that $M = H_1 + H_2$ is dense in $E$. If $(a_n)$ and $(b_n)$ are orthogonal bases of $H_1$ and $H_2$ respectively, $u$ is defined by taking $u(a_{2n-1}) = a_{2n}$, $u(a_{2n}) = -a_{2n-1}$, $u(b_{2n-1}) = b_{2n}$.

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$u(b_{2n}) = -b_{2n-1}$; and $L_n$ is the closed subspace of $E$ generated by $H_1$ and the vectors $b_i$ such that $1 \leq i \leq 2n$.

Such a situation excludes the possibility of proving the existence of a complex structure on a real Banach space by an inductive argument (of the type used, for instance, in the proof of Hahn-Banach's theorem). As was pointed out by the referee, the same example shows that a linear mapping of $M$ onto itself can be continuous on both subspaces $H_1$, $H_2$ without being continuous on $M$ itself.

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