# ON A LATTICE WITH A VALUATION

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A real-valued function $v(x)$ defined on a lattice is called a valuation if and only if it satisfies

$$v(x) + v(y) = v(x \cap y) + v(x \cup y), \tag{1}$$

and a distributive valuation if and only if it satisfies

$$\begin{aligned} 2\{v(x \cup y \cup z) - v(x \cap y \cap z)\} &= v(x \cup y) + v(y \cup z) \\ &+ v(z \cup x) - v(x \cap y) - v(y \cap z) - v(z \cap x). \end{aligned} \tag{2}$$

If $z = x \cup y$, then (2) becomes (1); therefore a distributive valuation is a valuation.

In his book *Lattice theory*, G. Birkhoff conjectured the following theorem.

THEOREM 1. *If $L$ is a lattice with a distributive valuation which is not constant on an interval $[x, y]$ unless $x = y$, then $L$ is distributive.*[1]

We intend to affirm this proposition by proving more precisely the following theorem.

THEOREM 2. *A lattice $L$ is distributive if and only if the following condition (*) is satisfied: (*) For every $x < y$ in $L$, there exists a distributive valuation which is defined on $L$ and is not constant on the interval $[x, y]$.*

We now begin with a lemma.

LEMMA 1. *A lattice $L$ is modular if, for every $x < y$ in $L$, there exists a valuation which is defined on $L$ and is not constant on the interval $[x, y]$.*

PROOF. If $L$ is not modular, then $L$ contains a nonmodular five-element sublattice, in which

$$a \cup b = a \cup c = e, \qquad a \cap b = a \cap c = f, \qquad b < c.$$

If $t$ is an element of the interval $[b, c]$, then every valuation $v(t)$ satisfies

$$v(t) + v(a) = v(t \cap a) + v(t \cup a) = v(f) + v(e);$$

hence $v(t)$ is constant on the interval $[b, c]$.

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[1] G. Birkhoff, *Lattice theory*, Amer. Math. Soc. Colloquium Publications, rev. ed., 1948, Problem 71.

LEMMA 2. *Let a modular lattice $L$ contain a nondistributive five-element sublattice $A$, in which*

$$a \cup b = b \cup c = c \cup a = e, \qquad a \cap b = b \cap c = c \cap a = f,$$

*and $t$ be an element of $L$ such that $a \leqq t < e$. Then $L$ contains a new nondistributive five-element sublattice $B$, in which*

$$s \cup t = t \cup r = r \cup s = e, \qquad s \cap t = t \cap r = r \cap s = g.$$

PROOF. Put $s=(t\cap c)\cup b$ and $r=(t\cap b)\cup c$. Then immediately $s\cup t=t\cup r=r\cup s=e$. As $L$ is modular, $s\cap t=((t\cap c)\cup b)\cap t=(t\cap c)\cup(b\cap t)$, and similarly $r\cap t=(t\cap b)\cup(c\cap t)$. The modular law also implies that $s\cap c=((t\cap c)\cup b)\cap c=(t\cap c)\cup(b\cap c)=t\cap c$, and $s\cap r=s\cap(c\cup(t\cap b))=(s\cap c)\cup(t\cap b)=(t\cap c)\cup(t\cap b)$. Consequently $s\cap t=t\cap r=r\cap s(=g)$. As $t\neq e$, the five elements of $B$ are mutually distinct.

LEMMA 3. *If the above five-element lattice $A$ is a sublattice of a lattice with a distributive valuation $v(x)$, then $v(a)=v(b)=v(c)=v(e)=v(f)$.*

PROOF. Using (1), we have $v(b)+v(c)=v(e)+v(f)$, $v(c)+v(a)=v(e)+v(f)$, and $v(a)+v(b)=v(e)+v(f)$, whence $v(a)=v(b)=v(c)=\{v(e)+v(f)\}/2$. If we put $x=a$, $y=b$, $z=c$ in (2), then we have $2\{v(e)-v(f)\}=3v(e)-3v(f)$, namely $v(e)=v(f)$.

PROOF OF THEOREM 2. Let $L$ be a lattice satisfying the condition (*). Then, by Lemma 1, $L$ is modular. If $L$ is not distributive, then $L$ contains a nondistributive five-element sublattice $A$ as mentioned in Lemma 2. Given an element $t$ $(t\neq e)$ of the interval $[a, e]$, there exists a nondistributive five-element sublattice $B$ as shown in Lemma 2. Then Lemma 3 implies that $v(t)=v(e)$ for every distributive valuation $v(x)$; namely every $v(x)$ is constant on the interval $[a, e]$. This contradicts the condition (*).

Conversely suppose that $L$ is a distributive lattice. For any $x<y$ in $L$, there exists a prime ideal $P$ which contains $x$ and not $y$. If we define a function $v(t)$ such that $v(t)=0$ for $t\in P$ and $v(t)=1$ for $t\notin P$, then $v(t)$ is a distributive valuation and $v(x)\neq v(y)$. Thus our theorem is completely proved.

By the way, all the statements in this note are valid even if $v(t)$ takes values in an Abelian group in which no element has the order 2.

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