MOMENTS WHICH ARE INTEGERS

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Let $\alpha(t)$ be of bounded variation and have infinitely many points of increase on $(0, a)$. *If* $\int_0^a t^n d\alpha(t)$ *is an integer for* $n=0, 1, 2, \cdots$, *then* $a \geqq 4$.

To prove this let I_n be the integral in the theorem, so that

$$f(z) = \sum I_n z^n = \int_0^a \frac{d\alpha(t)}{1 - zt} \cdot \tag{1}$$

The right side of (1) shows that $f(z)$ is regular except for $1/a \leqq z < \infty$ and this region has image-radius greater than 1 if $a<4$. The substitution $z=1/s$ turns (1) into a Hilbert transform (as was pointed out to the author by G. M. Wing) and the complex inversion formula shows then that $f(z)$ can be rational only when the integral reduces to a finite sum. The result now follows from the Pólya-Carlson theorem. Letting $t=4 \sin^2 \theta$ in

$$\pi I_n = \int_0^4 t^{n+1/2}(4 - t)^{-1/2} dt \tag{2}$$

shows that *the condition* $a \geqq 4$ *cannot be replaced by* $a>4$.

If $q(n)$ is a polynomial in n, the operator (zd/dz) applied to $(1-z)^{-1}$ shows that $\sum q(n) z^n$ has, as sole singularity, a pole at $z=1$. Hence *the condition* $I_n =$ *integer can be replaced by* $I_n = p_n/q(n)$ *where* p_n *are integers and* $q(n)$ *is a polynomial with integral coefficients.* Here we have the beginnings of a way to show that certain definite integrals (satisfying suitable recursion relations) are irrational. The discussion can be extended to $\int_0^a f_n(t) d\alpha(t)$, incidentally, where f_n has a known generating function $g(z, t) = \sum f_n(t) z^n$.

In case the limits of integration are (a, b), the conclusion of the theorem is $b-a \geqq 4$, as was pointed out by R. Steinberg. For integral a an example can be constructed after the manner of (2).

Open questions are: Is $b-a \geqq 4$ the optimum result for nonintegral a? How far can proofs of irrationality be carried by these methods? Is there a counter-example like (2) belonging to L^2 (or L^∞) on $(0, 4)$? Is the counter-example essentially unique? Must it always be simply related to the mapping function (as (2) is)?

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Received by the editors May 11, 1951.