MOMENTS WHICH ARE INTEGERS

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Let \( \alpha(t) \) be of bounded variation and have infinitely many points of increase on \((0, a)\). If \( \int_0^a t^n d\alpha(t) \) is an integer for \( n = 0, 1, 2, \cdots \), then \( a \geq 4 \).

To prove this let \( I_n \) be the integral in the theorem, so that

\[
(1) \quad f(z) = \sum I_n z^n = \int_0^a \frac{d\alpha(t)}{1 - zt}.
\]

The right side of (1) shows that \( f(z) \) is regular except for \( 1/a \leq z < \infty \) and this region has image-radius greater than 1 if \( a < 4 \). The substitution \( z = 1/s \) turns (1) into a Hilbert transform (as was pointed out to the author by G. M. Wing) and the complex inversion formula shows then that \( f(z) \) can be rational only when the integral reduces to a finite sum. The result now follows from the Pólya-Carlson theorem. Letting \( t = 4 \sin^2 \theta \) in

\[
(2) \quad \pi I_n = \int_0^1 t^{n+1/2}(4 - t)^{-1/2}dt
\]

shows that the condition \( a \geq 4 \) cannot be replaced by \( a > 4 \).

If \( q(n) \) is a polynomial in \( n \), the operator \((zd/dz)\) applied to \((1-z)^{-1}\) shows that \( \sum q(n) z^n \) has, as sole singularity, a pole at \( z = 1 \). Hence the condition \( I_n = \text{integer} \) can be replaced by \( I_n = p_n/q(n) \) where \( p_n \) are integers and \( q(n) \) is a polynomial with integral coefficients. Here we have the beginnings of a way to show that certain definite integrals (satisfying suitable recursion relations) are irrational. The discussion can be extended to \( \int f_n(t) d\alpha(t) \), incidentally, where \( f_n \) has a known generating function \( g(z, t) = \sum f_n(t) z^n \).

In case the limits of integration are \((a, b)\), the conclusion of the theorem is \( b - a \geq 4 \), as was pointed out by R. Steinberg. For integral \( a \) an example can be constructed after the manner of (2).

Open questions are: Is \( b - a \geq 4 \) the optimum result for nonintegral \( a \)? How far can proofs of irrationality be carried by these methods? Is there a counter-example like (2) belonging to \( L^2 \) (or \( L^n \)) on \((0, 4)\)? Is the counter-example essentially unique? Must it always be simply related to the mapping function (as (2) is)?