ON SOME FUNCTIONS HOLOMORPHIC IN AN INFINITE REGION

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S. Mandelbrojt indicated the following proposition: If a function is holomorphic and bounded in a half-strip of the $z$-plane containing the half-axis $ox$ as a part of its central line and if this function and a certain infinite sequence of its derivatives vanish at the origin, then it is identically zero. The proof of this proposition is based upon a result of Mandelbrojt [1, p. 372]. In the present paper, we consider a function $F(z)$ holomorphic in a region $A$ of the $z$-plane defined by $x \geq d$, $|y| \leq g(x)$, where $-\infty < d < 0$ and where $g(x)$ is a certain positive continuous function tending to zero with $1/x$. In this case if, in $A$, $F(z)$ tends to zero rapidly enough and uniformly with respect to $y$ as $x$ tends to infinity, and if $F(z)$ and a certain infinite sequence of its derivatives vanish at the origin, then $F(z)$ is identically zero. In order to establish our proposition, we prove at first a lemma by means of the following theorem of G. Valiron [3, p. 62, §32]:

**Theorem V.** Let $Y(X)$ be a real function having a first derivative for $X \geq X_0$ such that

$$\lim_{z \to \infty} \frac{XY'(X)}{\psi(X)} = 1; \quad \psi(X) \geq 1, \quad X \geq X_0; \quad \lim_{z \to \infty} \frac{X\psi(X)}{[\psi(X)]^2} = 0.$$

Let $\Phi(X)$ be an entire function and let $M(r) = \max_{|z|=r} |\Phi(z)|$. Then a necessary and sufficient condition that

Presented to the Society, September 7, 1951; received by the editors May 1, 1951.

1 The author wishes to express to Professors S. Mandelbrojt and G. Valiron his respectful gratitude for their kind and precious suggestions and criticisms.

2 Numbers in brackets refer to the bibliography at the end of this paper.
\[ \log M(r) \sim e^{Y'(X)} \quad X = \log r, \]

is that

\[ \nu(r) \sim Y'(X)e^{Y'(X)} \sim \frac{d}{dX} \log M(r), \]

where \( \nu(r) \) is the rank of the maximum term of the highest rank of the Taylor expansion of \( \Phi(z) \) corresponding to the value \( |z| = r \).

**Lemma.** Let \( \Phi(z) = \sum_{n=0}^{\infty} \phi(n)z^n \) and let \( \mu(r) \) be the value of the maximum terms of \( |\phi(n)|r^n \) \( (n = 0, 1, 2, \ldots) \). If

\[ \mu(r) \sim K[(\log_2 r)(\log_3 r) \cdots (\log_{p+1} r)]^{\log r} \quad (K = \text{const.} > 0), \]

then for any given \( \epsilon > 0 \) \( (\epsilon < 1) \), we have, for \( n \) sufficiently large,

\[ |\phi(n)| < \exp \{- \exp [\omega_p(e^{(1-\epsilon)n})]\} \]

and, for a sequence \( \{n_k\} \) such that \( n_{k+1}/n_k \) tends to 1 as \( k \) tends to infinity,

\[ |\phi(n_k)| > \exp \{- n_k \exp [\omega_p(e^{(1+\epsilon)n})]\} \]

where \( p \) is a positive integer and where \( \xi = \omega_p(\eta) \) is the inverse function of \( \eta = \xi(\log \xi)(\log_2 \xi) \cdots (\log_{p-1} \xi) \).

**Proof.** Since [3, p. 111 and 4, p. 32, chap. II]

\[ \log M(r) \sim \log \mu(r) \sim (\log r)(\log_2 r + \log_4 r + \cdots + \log_{p+1} r), \]

we have, by Theorem V,

\[ \nu(r) \sim \log [(\log_2 r)(\log_3 r) \cdots (\log_{p+1} r)]. \]

Considering with Valiron a polygon of Newton and using his notations, we see that

\[ n \sim \log [(\log_2 R_n)(\log_3 R_n) \cdots (\log_{p+1} R_n)]. \]

\( \omega_p(\eta) \) being an increasing function, it follows that

\[ \exp \{\omega_p(e^{(1-\epsilon)n})\} < e^{\alpha_n} = e^{\omega_p R_2 R_4 \cdots R_{p+1}} \]

\[ < \exp \{n \exp [\omega_p(e^{(1+\epsilon)n})]\} \]

for \( n \) sufficiently large. The lemma will then be completely established by Valiron's reasonings.

The following result is an immediate corollary of our lemma:

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\footnote{We write \( \log_k x = x \) and \( \log_k (x) = \log (\log_{k-1} x) \), \( k \) being a positive integer and \( x \) being sufficiently large.}
**Corollary.** If for a given $\epsilon > 0$,

$$\phi(n) = \exp \{-n \exp [\omega_p(e^{(1+\epsilon)n})]\}$$

for $n$ sufficiently large, then we have

$$\mu(r) \leq [(\log_3 r)(\log_2 r) \cdots (\log_{p+1} r)]^{\log_2 r}$$

for $r$ sufficiently large.

Now we can prove our main theorem:

**Theorem.** Let $g(x)$ be a positive continuous function defined for $x \geq d (-\infty < d < 0)$ decreasing to zero with $1/x$ for $x$ sufficiently large and satisfying

$$g(x) = O[g(x + \eta)] \quad (x \to \infty)$$

for $|\eta|$ sufficiently small. Denote by $\Delta$ the region of the $z$-plane defined by $x \geq d$, $|y| \leq g(x)$.

Let $\{n_1\}$ and $\{q_n\}$ be two complementary sequences of non-negative integers such that the upper density function $D^*(q)$ of $\{q_n\}$ satisfies, for $q$ sufficiently large,

$$D^*(q) < \frac{b}{(\log q)(\log_2 q) \cdots (\log_{p+1} q)} \quad \left(0 < b = \text{const.} < \frac{1}{2}\right).$$

Suppose that $F(z)$ is a function holomorphic in $\Delta$ and satisfying

$$F^{(n)}(0) = 0$$

and, for a given $\epsilon > 0$,

$$F(z) = O\left\{ [g(x)]^{\exp [\omega_p (e^{(1-\epsilon)n})]} \right\} \quad (z \in \Delta, x \to \infty).$$

Then we conclude $F(z) = 0$.

**Proof.** We can evaluate the moduli of all the derivatives of $F(z)$ on the half-axis $ox$: $x \geq 0$, $y = 0$. Let us put

$$h(x) = \min \{x - d, g(\xi)\} \quad [x \geq 0, \ |x - \xi| \leq g(x)]$$

and construct in the $z$-plane circles $C(x)$: $|z - x| \leq h(x)$ which are evidently situated in $\Delta$. We have

$$F^{(n)}(x) = \frac{n!}{2\pi i} \int_{C(x)} \frac{F(z)}{(z - x)^{n+1}} \, dz \quad (x \geq 0).$$

By hypotheses there exist positive constants $A$, $B$, $E$ and $x_0 > d$ such that:
Hence we obtain
\[ \Omega_n(x, g(x)) \leq [K_2(\log n)(\log_2 n) \cdots (\log_p n)]^n \quad (K_2 = \text{const.} > 0) \]
for \( x \geq x_0 - e^{-\alpha} \) and for \( n \) sufficiently large. (We pass from the case \( g(x) = e^{-x} \) to the general case simply by replacing \( e^{-x} \) in what precedes by \( g(x) \).) Consequently we have
\[ |F^{(n)}(x)| \leq [K_3(\log n)(\log_2 n) \cdots (\log_p n)]^n \quad (K_3 = \text{const.} > 0) \]
for \( x \geq 0 \) and for \( n \) sufficiently large. \( F(x) \) and its derivatives of lower
orders are evidently also bounded for \( x \geq 0 \). An application of a Mandelbrojt’s result on generalized quasi-analyticity [2, chap. III] will complete immediately the proof of our theorem.

From this theorem it follows that if \( F_1(z) \) and \( F_2(z) \) are functions holomorphic in \( \Delta \) and verifying conditions similar to (4) and if \( F_1^{(n)}(0) = F_2^{(n)}(0) \) for a sequence \( \{v_n\} \) defined in the above theorem, then we have \( F_1(z) = F_2(z) \).

We remark that in the case \( p = 1 \), (4) reduces to

\[
F(z) = O\left( \left[ g(x) \right]^{\exp \{ \nu(z)^{-1-1} \}} \right) \quad (z \text{ in } \Delta; x \to \infty).
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Bibliography


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\*For the case \( p = 1 \) of the mentioned result, see [1, p. 372].