MEAN CONVERGENCE OF ORTHOGONAL SERIES
JEROME NEWMAN AND WALTER RUDIN

I. Introduction. Marcel Riesz's theorem [9, p. 153] about the mean convergence of Fourier series in $L^p$ spaces has been extended by Harry Pollard to series of classical orthogonal polynomials [2; 3; 4; 5]. In particular, it was shown that the Legendre expansion of a function $f \in L^p$ on $(-1, 1)$ converges in the mean of order $p$ to $f$ if $4/3 < p < 4$, but that this is false for any $p > 4$ (or $1 \leq p < 4/3$). The question of the status of the limiting values $p = 4$ and $p = 4/3$ was raised but not settled by Pollard, and is repeated in the reviews of [3] and [5].

Entirely similar results were established for Jacobi series (for the limiting values of $p$, see the statements of Theorems 2 and 3 below), and for series of Bessel functions (Theorem 4), where the problem was also left open for $p = 4/3$ and $p = 4$ [8, pp. 797 and 802].

The purpose of the present paper is to show by a very simple argument that in all the above cases the property of mean convergence fails to hold at the limiting values of $p$.

II. A necessary condition. Suppose the sequence $\{\phi_n(x)\}$ is orthonormal with respect to a measure function $\mu(x)$ on $(a, b)$, that is,

$$\int_a^b \phi_n(x)\phi_m(x)d\mu(x) = \delta_{mn}.$$ 

Let $L^p_\mu$ denote the space of functions $f$ for which

$$\|f\|_{p, \mu} = \left\{\int_a^b \left| f(x) \right|^pd\mu(x) \right\}^{1/p} < \infty \quad (p \geq 1),$$

and put

$$(1) \quad a_n(f) = \int_a^b f(x)\phi_n(x)d\mu(x).$$

Then if the series

$$(2) \quad \sum_{n=0}^\infty a_n(f)\phi_n(x)$$
converges in $L^p_\mu$ for every $f \in L^p_\mu$, we must have
\begin{equation}
 a_n(f) \| \phi_n \|_{p,\mu} = O(1)
\end{equation}
for every $f \in L^p_\mu$. Hence the Banach-Steinhaus theorem \cite[Theorem 5]{1} applied to the functionals (3) shows:

**Theorem 1.** If the series (2) converges in $L^p_\mu$ for every $f \in L^p_\mu$, then
\begin{equation}
 \| \phi_n \|_{p,\mu} \| \phi_n \|_{q,\mu} = O(1) \quad \left( \frac{1}{p} + \frac{1}{q} = 1 \right).
\end{equation}

IIII. Jacobi series and Bessel series. Let $\alpha \geq -1/2$, $\beta \geq -1/2$, be fixed. The polynomials $P_n^{(\alpha,\beta)}(x)$ (we use Szegö’s notation \cite[p. 57]{6}) are orthogonal on $(-1, 1)$ with respect to the weight function
\[ w(x) = (1 - x)^\alpha (1 + x)^\beta. \]
If the sequence $\{c_n\}$ is determined by the conditions
\[ \int_{-1}^{1} [c_n P_n^{(\alpha,\beta)}(x)]^2 w(x) dx = 1 \quad (n = 0, 1, 2, \ldots ; c_n > 0), \]
we put
\[ \Phi_n(x) = c_n P_n^{(\alpha,\beta)}(x), \]
\[ \Psi_n(x) = c_n P_n^{(\alpha,\beta)}(x) [w(x)]^{1/2}. \]
It is known that $c_n$ is of order $n^{1/2}$ \cite[formula (4.3.3)]{5}.

We suppose that $\alpha \geq \beta$ (the results are completely analogous if $\alpha < \beta$), and put
\[ r = \frac{4(\alpha + 1)}{2\alpha + 3}, \quad s = \frac{4(\alpha + 1)}{2\alpha + 1}. \]

**Theorem 2.** If $p = r$, or if $p = s$, there exists a function $f \in L^p_\mu$ on $(-1, 1)$ such that the series $\sum a_n(f) \Phi_n(x)$ does not converge in $L^p_\mu$.

Here
\[ a_n(f) = \int_{-1}^{1} f(x) \Phi_n(x) w(x) dx. \]

**Theorem 3.** If $p = 4/3$, or if $p = 4$, there exists a function $f \in L^p$ on $(-1, 1)$ such that the series $\sum a_n(f) \Psi_n(x)$ does not converge in $L^p$. 

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
Here

\[ a_n(f) = \int_{-1}^{1} f(x) \psi_n(x) \, dx. \]

Next, let \( \xi \) be real, and \( \nu \geq -1/2 \). Let \( 0 < \lambda_1 < \lambda_2 < \cdots \) be the positive real zeros of the function \( J_\nu(x) \cos \xi + x J'_\nu(x) \sin \xi \). The functions \( J_\nu(\lambda_n x) \) \((n = 1, 2, 3, \cdots)\) are orthogonal on \((0, 1)\) with respect to the weight function \( x \). If the sequence \( \{b_n\} \) is determined by the conditions

\[ \int_{0}^{1} [b_n J_\nu(\lambda_n x)]^2 x \, dx = 1 \quad (n = 1, 2, 3, \cdots; b_n > 0), \]

we put

\[ B_n(x) = b_n J_\nu(\lambda_n x). \]

Let us note that the numbers \( \lambda_n \) are interlaced with the positive zeros of \( J_\nu(x) \) \([7, \text{p. 480}]\). Since the latter are of order \( n \) \([7, \text{p. 506}]\), so is \( \lambda_n \). It follows that \( b_n \) is of order \( n^{1/2} \) \([7, \text{p. 577}]\).

**Theorem 4.** If \( p = 4/3 \), or if \( p = 4 \), there exists a function \( f \in L_p^p \) on \((0, 1)\) \((d\mu(x) = x \, dx)\) such that the series \( \sum a_n(f) B_n(x) \) does not converge in \( L_p^p \).

**Proofs.** For Theorems 2 and 3 we use the formula \([5, (8.21.18)]\)

\[ P_n^{(\alpha, \beta)}(\cos \theta) = n^{-1/2} k(\theta) \left\{ \cos \left( N\theta + \gamma \right) + (n \sin \theta)^{-1} O(1) \right\}, \]

where \( N = n + (\alpha + \beta + 1)/2, \quad \gamma = -(2\alpha + 1)\pi/4, \) and \( k(\theta) = \pi^{-1/2} \cdot (\sin (\theta/2))^{-1/4} \cdot (\cos (\theta/2))^{-\theta - 1/2}. \)

Putting \( x = \cos \theta \), and applying (5) on the interval \([n^{-1}, \pi - n^{-1}]\) on which the error term is uniform, a simple calculation shows that

\[ \left\{ \int_{-1}^{1} | \Phi_n(x) |^r w(x) \, dx \right\}^{1/r} > A_1 (\log n)^{1/r}, \]

\[ \left\{ \int_{-1}^{1} | \Phi_n(x) |^r w(x) \, dx \right\}^{1/r} > A_3, \]

\[ \left\{ \int_{-1}^{1} | \psi_n(x) \, dx \right\}^{1/4} > A_3 (\log n)^{1/4}, \]

\[ \left\{ \int_{-1}^{1} | \psi_n^4(x) \, dx \right\}^{1/4} > A_4. \]

Here \( A_1, A_3, \cdots \) are positive constants. Theorems 2 and 3 fol-
low since (4) is violated in each case.

It should be noted that (6) was derived on the tacit assumption that \( s < \infty \). If \( s = \infty \), then \( \alpha = -1/2 \), \( P^{(\alpha \beta)}_n \) is a Tchebichef polynomial, and Theorem 2 reduces to a statement about trigonometric series which is well known [9, p. 155].

Finally, to prove Theorem 4, we put \( y = \lambda_n x \) in the integrals involving \( J_\nu(\lambda_n x) \), use the asymptotic formula [7, p. 199]

\[
J_\nu(y) \sim \left( \frac{2}{\pi y} \right)^{1/2} \cos \left( y - \frac{\nu \pi}{2} - \frac{\pi}{4} \right) \quad (y \to \infty),
\]

and find that

\[
\left\{ \int_0^1 B_n(x) x \, dx \right\}^{1/4} > A_n (\log n)^{1/4},
\]

(10)

\[
\left\{ \int_0^1 |B_n(x)|^{4/3} x \, dx \right\}^{3/4} > A_n.
\]

(11)

Thus (4) is again violated.

**Bibliography**